10

State-Variable Techniques

K. S. Chao
Texas Tech University

10.1 The Concept of States

For resistive (or memoryless) circuits, given the circuit structure, the present output depends only on the present input. In order to analyze a dynamic circuit, however, in addition to the present input it is also necessary to know the state of the circuit at some time t_0. The state of the circuit at t_0 represents the condition of the circuit at $t = t_0$, and is related to the energy storage of the circuit, or the voltage (or electric charge) across the capacitor and the currents (or magnetic fluxes) through the inductors. These voltages and currents are considered as the state of the circuit at $t = t_0$. For $t > t_0$, the behavior of the circuit is completely characterized by these variables. In view of the preceding, a definition for the state of a circuit can now be given.

Definition: The state of a circuit at time t_0 is the minimum amount of information at t_0 that, along with the input to the circuit for $t \geq t_0$, uniquely determines the behavior of the circuit for $t \geq t_0$.

The concept of states is closely related to the order of complexity of the circuit. The order of complexity of a circuit is the minimum number of initial conditions which, along with the input, is sufficient to determine the future behavior of the circuit. Furthermore, if a circuit is described by an n^{th}-order linear differential equation, it is well known that the general solution for $t \geq t_0$ contains n arbitrary constants which are determined by n initial conditions. This set of n initial conditions contains information concerning the circuit prior to $t = t_0$ and constitutes the state of the circuit at $t = t_0$. Thus, the order of complexity or the order of a circuit is the same as the order of the differential equation that describes the circuit, and it is also the same as the number of state variables that can be defined in a circuit. For an n^{th}-order circuit, the state of the circuit at $t = t_0$ consists of a set of n numbers that denotes a vector in an n-dimensional state space spanned by the n corresponding state variables. This key number n can simply be obtained by inspection of the circuit. Knowing the total number of energy storage elements, n_{LC}, the total number of independent capacitive loops, n_C, and the total number of independent inductive cutsets, n_L, the order of complexity n of a circuit is given by

$$n = n_{LC} - n_L - n_C \tag{10.1}$$

A capacitive loop is defined as one that consists of only capacitors and possibly voltage sources while an inductive cutset represents a cutset that contains only inductors and possibly current sources. The following two examples illustrate the concept of states.

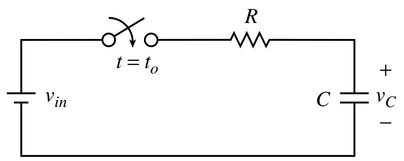

FIGURE 10.1 A simple RC circuit.

Example 1. Consider a simple RC-circuit in Figure 10.1. The circuit equation is

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_{in} \qquad \text{for} \qquad t\geq t_0 \tag{10.2}$$

and the corresponding capacitor voltage is easily obtained as

$$v_c(t)=\left[v_c(t_0)-v_{in}\right]e^{-\frac{1}{RC}(t-t_0)}+v_{in} \quad \text{for} \qquad t\geq t_0 \tag{10.3}$$

For this first-order circuit, it is clear from (10.3) the capacitor voltage for $t \geq t_0$ is uniquely determined by the initial condition $v_c(t_0)$ and the input voltage v_{in} for $t \geq t_0$. This is independent of the charging circuit for the capacitor prior to t_0. Hence, $v_c(t_0)$ is the state of the circuit at $t = t_0$ and $v_c(t)$ is regarded as the state variable of the circuit.

Example 2. As another illustration, consider the circuit of Figure 10.2, which is a slight modification of the circuit considered in the previous example. The circuit equation and its corresponding solution are readily obtained as

$$\frac{dv_{C_1}(t)}{dt}=-\frac{1}{R(C_1+C_2)}v_{C_1}(t)+\frac{1}{R(C_1+C_2)}v_{in} \tag{10.4}$$

and

$$v_{C_1}(t)=\left(v_{C_1}(t_0)-v_{in}\right)e^{\frac{-1}{R(C_1+C_2)}(t-t_0)}+v_{in} \quad \text{for} \qquad t\geq t_0 \tag{10.5}$$

respectively. Even though two energy storage elements exist, one can only arbitrarily specify one independent initial condition. Once the initial condition on C_1, $v_{C_1}(t_0)$, is specified, the initial voltage on C_2 is automatically constrained by the loop equation $v_{C_2}(t) = V_{C_1}(t) - E$ at t_0. The circuit is thus still first order and only one state variable can be assigned for the circuit. It is clear from (10.5) that with the input v_{in}, $v_{C_1}(t_0)$ is the minimum amount of information that is needed to uniquely determine the behavior of this circuit. Hence, $v_{C_1}(t)$ is the state variable of the circuit. One can just as well analyze the circuit by solving a first-order differential equation in terms of $v_{C_2}(t)$ with $v_{C_2}(t_0)$ defined as the state of the circuit at $t = t_0$. The selection of state variables is thus not unique. In this example, either $v_{C_1}(t)$ or $v_{C_2}(t)$ can be defined as the state variable of the circuit. In fact, it is easily shown that any linear combination of $v_{C_1}(t)$ and $v_{C_2}(t)$ can also be regarded as state variables.

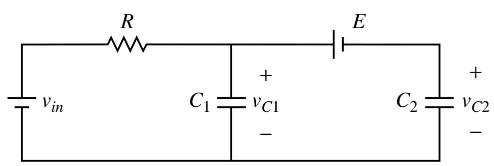

FIGURE 10.2 The circuit for Example 2.

10.2 State-Variable Formulation via Network Topology

Various mathematical descriptions of circuits are available. Depending on the type of analysis used, different formulations of circuit equations may result. In the state variable formulation, a system of n first-order differential equations is written in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{10.6}$$

where $\mathbf{x}$ is an n x 1 vector consisting of n state variables for an n^{th}-order circuit and t represents the time variable. This set of equations is usually referred to as the state equation in normal form.

When compared with other circuit descriptions, the state-variable representation is not necessarily the simplest. It does, however, simultaneously provide the solution of all state variables and hence yields the behavior of the entire circuit. The state equation is also particularly suitable for analysis by numerical techniques. Another distinct advantage of the state-variable approach is that it can be easily extended to nonlinear and/or time varying circuits.

Example 3. Consider the linear circuit of Figure 10.3. By inspection, the order of complexity of this circuit is three. Hence, three state variables are selected as $x_1 = v_{C1}$, $x_2 = v_{C2}$, and $x_3 = i_L$. Because the left-hand side of the normal form equation is the derivative of the state vector, it is necessary to express the voltage across the inductors and the currents through the capacitors in terms of the state variables and the input sources.

The current through C_1 can be obtained by writing a Kirchhoff's current law (KCL) equation at node 1 to yield

$$\begin{aligned} C_1 \frac{dv_{C_1}}{dt} &= i_{R_1} - i_L \\ &= \frac{1}{R_1}\left(v_s - v_{C_1}\right) - i_L \end{aligned}$$

or

$$\frac{dv_{C_1}(t)}{dt} = -\frac{1}{R_1 C_1} v_{C_1} - \frac{1}{C_1} i_L + \frac{1}{R_1 C_1} v_s \tag{10.7}$$

In a similar manner, applying KCL to node 2 gives

$$\begin{aligned} C_2 \frac{dv_{C_2}}{dt} &= i_L - i_{R_3} + i_s \\ &= i_L - \frac{1}{R_3} v_{C_2} + i_s \end{aligned}$$

or

$$\frac{dv_{C_2}(t)}{dt} = -\frac{1}{R_3 C_2} v_{C_2} + \frac{1}{C_2} i_L + \frac{1}{C_2} i_s \tag{10.8}$$

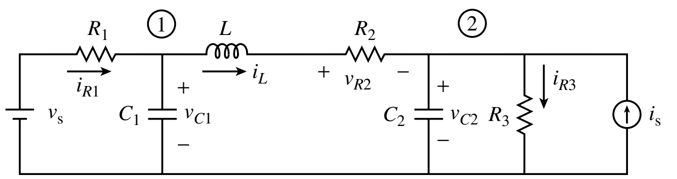

FIGURE 10.3 The circuit for Example 3.

The expression for the inductor voltage is derived by applying KVL to the mesh containing L, R_2, C_2, and C_1 yielding

$$L\frac{di_L}{dt} = v_{C_1} - v_{C_2} - R_2 i_L$$

or

$$\frac{di_L}{dt} = \frac{1}{L}v_{C_1} - \frac{1}{L}v_{C_2} - \frac{R_2}{L}i_L \tag{10.9}$$

Equations (10.7), (10.8), and (10.9) are the state equations that can be expressed in matrix form as

$$\begin{bmatrix} \frac{dv_{C_1}}{dt} \\ \frac{dv_{C_2}}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ 0 & -\frac{1}{R_3C_2} & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} v_{c_1} \\ v_{c_2} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ 0 & \frac{1}{C_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix} \tag{10.10}$$

Any number of branch voltages and/or currents may be chosen as output variables. If i_{R_1} and v_{R_2} are considered as outputs for this example, then the output equations, written as a linear combination of state variables and input sources become

$$i_{R_1} = \frac{1}{R_1}\left(v_s - v_{C_1}\right) \tag{10.11}$$

$$v_{R_2} = R_2 i_L \tag{10.12}$$

or in matrix form

$$\begin{bmatrix} i_{R_1} \\ v_{R_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 0 & 0 \\ 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix} \tag{10.13}$$

In general, for an n^{th}-order linear circuit with r input sources and m outputs, the state and output equations are represented by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{10.14}$$

and

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{10.15}$$

where $\mathbf{x}$ is an $n \times 1$ state vector, $\boldsymbol{u}$ is an $r \times 1$ vector representing the r input sources, $m \times 1$-vector $\mathbf{y}$ denotes the m output variables, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are of order $n \times n$, $n \times r$, $m \times n$, and $m \times r$, respectively.

In the preceding example, the state equations are obtained by inspection for a simple circuit by writing voltage equations for inductors and current equations for capacitors and properly eliminating the non-state variables. For more complicated circuits, a systematic procedure for eliminating the nonstate variables is desirable. Such a procedure can be generated with the aid of a proper tree. A proper tree is a tree obtained from the associated network graph that contains all capacitors, independent voltage sources, and possibly some resistive elements, but does not contain inductors and independent current sources.

The selection of such a tree is always possible if the circuit contains no capacitive loops and no inductive cutsets. The reason for providing such a tree for writing state equations is obvious. With each tree branch, there is a unique cutset known as the fundamental cutset that contains only one tree branch and some links. Thus, if capacitors are in the tree, a fundamental cutset equation may be written for the corresponding currents through the capacitors. Similarly, every link (together with some tree branches) forms a unique loop called a fundamental loop. If inductors are selected as links, inductor voltages may be obtained by writing the corresponding fundamental loop equations. With the selection of a proper tree, state variables can be defined as the capacitor tree-branch voltages and inductive link currents. In view of the above observation, a systematic procedure for writing state equations can now be stated as follows:

STEP 1: From the associated directed graph, pick a proper tree.

STEP 2: Write fundamental cutset equations for the capacitive tree branches and express the capacitor currents in terms of link currents.

STEP 3: Write fundamental loop equations for the inductive links and express the inductor voltages in terms of tree-branch voltages.

STEP 4: Define the state variables. Capacitive tree-branch voltages and inductive link currents are selected as state variables. Other quantities such as capacitor charges and inductor fluxes may also be used.

STEP 5: Group the branch relations and the remaining fundamental equations according to their element types into three sets: resistor, inductor, and capacitor equations. Solve for the nonstate variables that appeared in the equations obtained in Steps 2 and 3 from the corresponding set of equations in terms of the state variables and independent sources.

STEP 6: Substitute the result of Step 5 into the equations obtained in Steps 2 and 3, and rearrange them in normal form.

Example 4. Consider again the same circuit in Figure 10.3. The various steps outlined previously are used to write the state equations.

STEP 1: The associated graph and the proper tree of the circuit are shown in Figure 10.4. The tree branches include v_s, C_1, C_2, and R_2.

STEP 2: The fundamental cutset associated with C_1 consists of tree branch C_1 and two links R_1 and L. By writing the current equation for this cutset, the capacitor current i_{c_1} is expressed in terms of link currents as

$$i_{C_1} = i_{R_1} - i_L \tag{10.16}$$

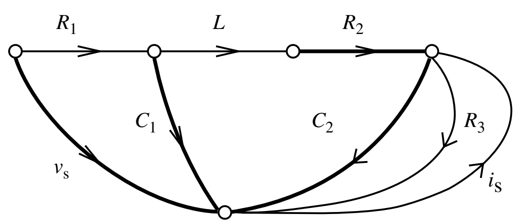

FIGURE 10.4 The directed graph associated with the circuit of Figure 10.3.

Similarly, the fundamental cutset $\{L, C_2, R_3, i_s\}$ associated with C_2 leads to

$$i_{C_2} = i_L - i_{R_3} + i_s \tag{10.17}$$

STEP 3: The fundamental loop associated with link L consists of L and tree branches R_2, C_2, and C_1. By writing the voltage equation around this loop, the inductor voltage can be written in terms of tree-branch voltages as

$$v_L = v_{C_1} - v_{C_2} - v_{R_2} \tag{10.18}$$

STEP 4: The tree-branch capacitor voltages v_{C1}, V_{C2}, and inductive link current i_L are defined as the state variables of the circuit.

STEP 5: The branch relation and the remaining two fundamental loops for R_1 and R_2, and the fundamental cutset equation for R_2 are grouped into three sets.

Resistor equations:

$$v_{R_1} + v_{C_1} - v_s = 0 \tag{10.19}$$

$$i_{R_1} = \frac{1}{R_1} v_{R_1} \tag{10.20}$$

$$i_{R_2} - i_L = 0 \tag{10.21}$$

$$v_{R_2} = R_2 i_{R_2} \tag{10.22}$$

$$v_{R_3} - v_{C_2} = 0 \tag{10.23}$$

$$v_{R_3} = R_3 i_{R_3} \tag{10.24}$$

Inductor equations:

$$\phi_L = L i_L \quad \text{or} \quad v_L = \frac{d\phi_L}{dt} = L\frac{di_L}{dt} \tag{10.25}$$

Capacitor equations:

$$q_1 = C_1 v_{C_1} \quad \text{or} \quad i_{C_1} = \frac{dq_1}{dt} = C_1 \frac{dv_{C_1}}{dt} \tag{10.26}$$

$$q_2 = C_2 v_{C_2} \quad \text{or} \quad i_{C_2} = \frac{dq_2}{dt} = C_2 \frac{dv_{C_2}}{dt} \tag{10.27}$$

The resistive link currents r_{R1}, i_{R3}, and resistive tree-branch voltage V_{R2} are solved from (10.19)–(10.24) in terms of the inductive link current i_L, the capacitive tree-branch voltages v_{C_1} and v_{C2}, and sources as

$$i_{R_1} = \frac{1}{R_1}\left(v_s - v_{C_1}\right) \tag{10.28}$$

$$i_{R_3} = \frac{1}{R_3} v_{C_2} \tag{10.29}$$

and

$$v_{R_2} = R_2 i_L \tag{10.30}$$

For this example, i_L, v_{C_1}, and v_{C_2} have already been defined as state variables.

STEP 6: Substituting (10.28)–(10.30) into (10.16), (10.17), and (10.18) yields the desired state equation in matrix form:

$$\begin{bmatrix} \dfrac{dv_{C_1}}{dt} \\ \dfrac{dv_{C_2}}{dt} \\ \dfrac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1C_1} & 0 & -\dfrac{1}{C_1} \\ 0 & -\dfrac{1}{R_3C_2} & \dfrac{1}{C_2} \\ \dfrac{1}{L} & -\dfrac{1}{L} & -\dfrac{R_2}{L} \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_L \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1C_1} & 0 \\ 0 & \dfrac{1}{C_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix} \tag{10.31}$$

which, as expected, is the same as (10.10) obtained previously by inspection.

As mentioned earlier, the selection of state variables is not unique. Instead of using capacitor voltages and inductor currents as state variables, basic quantities such as the capacitor charges and inductor fluxes may also be considered. If q_1, q_2, and ϕ_L are defined as state variables in Step 4, the inductive link current i_L and capacitive tree-branch voltages, v_{C_1} and v_{C_2}, can be solved from the inductor and capacitor equations in terms of state variables and possibly sources in Step 5 as

$$i_L = \frac{1}{L}\phi_L \tag{10.32}$$

$$v_{C_1} = \frac{1}{C_1} q_1 \tag{10.33}$$

$$v_{C_2} = \frac{1}{C_2} q_2 \tag{10.34}$$

Finally, state equations are obtained by substituting Eqs. (10.28)–(10.30) and (10.32)–(10.34) into (10.16)–(10.18) as

$$\begin{bmatrix} \dfrac{dq_1}{dt} \\ \dfrac{dq_2}{dt} \\ \dfrac{d\phi_L}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1C_1} & 0 & -\dfrac{1}{L} \\ 0 & -\dfrac{1}{R_3C_2} & \dfrac{1}{L} \\ \dfrac{1}{C_1} & -\dfrac{1}{C_2} & -\dfrac{R_2}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \Phi_L \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix} \tag{10.35}$$

In the systematic procedure outlined previously, it is assumed that the network exists with neither inductive cutsets nor capacitive loops so that the selection of proper tree is always guaranteed. For networks that do have these constraints, it is not possible to include all the capacitors in a tree without forming a closed path. Also, in order for a tree to contain all the nodes, some inductors will have to be included in a tree. A tree that includes independent voltage sources, some resistors, and a maximum number of capacitors but no independent current sources is called a modified proper tree. In writing a state equation for such networks, the same systematic procedure can be applied with the selection of a modified proper tree. However, if capacitor tree-branch voltages and inductive link currents are defined as the state variables, the standard (**A, B, C, D**) description (10.14) and (10.15) may not exist. In fact, if inductive cutsets contain independent current sources and/or capacitive loops contain independent voltage sources, the derivative of these sources will appear in the state equation and the general equation is of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\dot{\mathbf{u}} \tag{10.36}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are $n \times r$ matrices and $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{u}$ are defined as before. To recast (10.36) into the standard form, it is necessary to redefine.

$$\mathbf{z} = \mathbf{x} - \mathbf{B}_2\mathbf{u} \tag{10.37}$$

as new state variables. Substituting (10.37) into (10.36), yields

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \tag{10.38}$$

where

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{A}\mathbf{B}_2 \tag{10.39}$$

It is noted from (10.37), the new state variables represent a linear combination of sources and capacitor voltages or inductor currents which, except for the mathematical convenience, may not have sound physical significance. To avoid such state variables and transformation (10.37), Step 4 of the systematic procedure described earlier needs to be modified. By defining state variables as the algebraic sum of capacitor charges in the fundamental cutset associated with each of the capacitor tree branches, and the algebraic sum of inductor fluxes in the fundamental loop associated with each of the inductive links, the resulting state equation will be in the standard form. The preceding generalizations are illustrated by the following two examples.

Example 5. As a simple illustration, consider the same circuits given in Figure 10.2, where the constant DC voltage source E is replaced by a time-varying source $e(t)$. It can easily be demonstrated that the equation describing the circuit now becomes

$$\frac{dv_{C_1}(t)}{dt} = -\frac{1}{R(C_1+C_2)}v_{C_1} + \frac{1}{R(C_1+C_2)}v_{in}(t) + \frac{C_2}{R(C_1+C_2)}\frac{de(t)}{dt} \tag{10.40}$$

The preceding equation is the same as the state Eq. (10.4) with the exception of an additional term involving the first-order derivative of source $e(t)$. Equation (10.40) is clearly not the standard state equation described in (10.41) with capacitor voltage v_{C_1} defined as the state variable.

Example 6. As another illustration, consider the circuit shown in Figure 10.5 which consists of an inductive cutset $\{L_1, L_2, i_s\}$ and a capacitive loop (C_1, v_{s_2}, C_2). The state equations are determined from the systematic procedure by first using the transformation (10.37) and then by defining the algebraic sum of charges and fluxes as state variables.

STEP 1: The directed graph of the circuit is shown in Figure 10.6 where branches denoted by v_{s_1}, v_{s_2}, C_1, R_2, and L_2 are selected to form a modified proper tree.

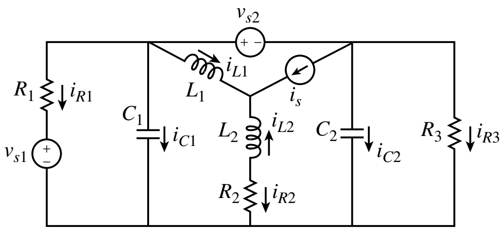

FIGURE 10.5 A circuit with a capacitive loop and an inductive cutset.

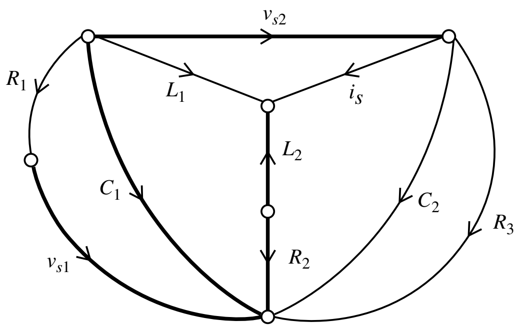

FIGURE 10.6 The directed graph associated with the circuit of Figure 10.5.

STEP 2: The fundamental cutset associated with C_1 consists of branches R_1, C_1, L_1, i_s, C_2, and R_3. Applying KCL to this cutset yields

$$i_{C_1} = -i_{R_1} - i_{L_1} - i_s - i_{C_2} - i_{R_3} \tag{10.41}$$

STEP 3: The fundamental loop equation associated with the inductive link L_1 is given by

$$v_{L_1} = v_{C_1} + v_{L_2} - v_{R_2} \tag{10.42}$$

where the link voltage v_{L_1} has been expressed in terms of tree-branch voltages.

STEP 4: In the first illustration, the tree-branch capacitor voltage v_{C_1} and the inductive link current i_{L_1} are defined as the state variables.

STEP 5: The branch relation and the remaining two fundamental equations are grouped into the following three sets:

Resistor equations:

$$v_{R_1} + v_{s_1} - v_{C_1} = 0 \tag{10.43}$$

$$v_{R_1} = R_1 i_{R_1} \tag{10.44}$$

$$i_{R_2} - i_{L_1} - i_s = 0 \tag{10.45}$$

$$v_{R_2} = R_2 i_{R_2} \tag{10.46}$$

$$v_{R_3} - v_{C_1} + v_{s_2} = 0 \tag{10.47}$$

$$v_{R_3} = R_3 i_{R_3} \tag{10.48}$$

Inductor equations:

$$i_{L_2} + i_{L_1} + i_s = 0 \tag{10.49}$$

$$\phi_{L_1} = L_1 i_{L_1} \quad \text{or} \quad v_{L_1} = L_1 \frac{di_{L_1}}{dt} \tag{10.50}$$

$$\phi_{L_2} = L_2 i_{L_2} \quad \text{or} \quad v_{L_2} = L_2 \frac{di_{L_2}}{dt} \tag{10.51}$$

Capacitor equations:

$$v_{C_2} - v_{C_1} + v_{s_2} = 0 \tag{10.52}$$

$$q_1 = C_1 v_{C_1} \quad \text{or} \quad i_{C_1} = C_1 \frac{dv_{C_1}}{dt} \tag{10.53}$$

$$q_2 = C_2 v_{C_2} \quad \text{or} \quad i_{C_2} = C_2 \frac{dv_{C_2}}{dt} \tag{10.54}$$

For this example, the nonstate variables are identified as i_{R1}, v_{R2}, i_{R3}, v_{L2}, and i_{C2}, from (10.41) and (10.42). These variables are now solved from the corresponding group of equations in terms of state variables and independent sources:

$$i_{R_1} = \frac{1}{R_1}\left(v_{C_1} - v_{s_1}\right) \tag{10.55}$$

$$v_{R_2} = R_2\left(i_{L_1} + i_s\right) \tag{10.56}$$

$$i_{R_3} = \frac{1}{R_3}\left(v_{C_1} - v_{s_2}\right) \tag{10.57}$$

$$v_{L_2} = -L_2 \frac{di_{L_1}}{dt} - L_2 \frac{di_s}{dt} \tag{10.58}$$

$$i_{C_2} = C_2 \frac{dv_{C_1}}{dt} - C_2 \frac{dv_{s_2}}{dt} \tag{10.59}$$

STEP 6: Assuming the existence of the first-order derivatives of sources with respect to time and substituting eqs. (10.50), (10.53), and (10.55)–(10.59) into (10.41) and (10.42) yields

$$\begin{bmatrix} \dfrac{dv_{C_1}}{dt} \\ \dfrac{di_{L_1}}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1+R_3}{R_1R_3(C_1+C_2)} & -\dfrac{1}{C_1+C_2} \\ \dfrac{1}{L_1+L_2} & -\dfrac{R_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} v_{C_1} \\ i_{L_1} \end{bmatrix}$$

$$+\begin{bmatrix} \dfrac{1}{R_1(C_1+C_2)} & \dfrac{1}{R_3(C_1+C_2)} & -\dfrac{1}{(C_1+C_2)} \\ 0 & 0 & -\dfrac{R_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} v_{s_1} \\ v_{s_2} \\ i_s \end{bmatrix} \tag{10.60}$$

$$+\begin{bmatrix} 0 & \dfrac{C_2}{(C_1+C_2)} & 0 \\ 0 & 0 & -\dfrac{L_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} \dfrac{dv_{s_1}}{dt} \\ \dfrac{dv_{s_2}}{dt} \\ \dfrac{di_s}{dt} \end{bmatrix}$$

Clearly, Eq. (10.60) is not in the standard form. Applying transformation (10.37) with $x_1 = v_{c1}$, $x_1 = i_{L2}$, $u_1 = v_{s_1}$, $u_2 = v_{s_2}$, and $u_3 = i_s$ gives the state equation in normal form

$$\begin{bmatrix} \dfrac{dz_1}{dt} \\ \dfrac{dz_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1+R_2}{R_1R_3(C_1+C_2)} & -\dfrac{1}{C_1+C_2} \\ \dfrac{1}{L_1+L_2} & -\dfrac{R_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$+\begin{bmatrix} \dfrac{1}{R_1(C_1+C_2)} & \dfrac{R_1C_1-R_3C_2}{R_1R_3(C_1+C_2)^2} & -\dfrac{L_1}{(L_1+L_2)(C_1+C_2)} \\ 0 & \dfrac{C_2}{(L_1+L_2)(C_1+C_2)} & -\dfrac{R_2L_1}{(L_1+L_2)^2} \end{bmatrix} \tag{10.61}$$

$$\times \begin{bmatrix} v_{s_1} \\ v_{s_2} \\ i_s \end{bmatrix}$$

where new state variables are defined as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} v_{C_1} & -\dfrac{C_2}{C_1+C_2}v_{s_2} \\ i_{L_1} & +\dfrac{L_2}{L_1+L_2}i_s \end{bmatrix} \tag{10.62}$$

Alternatively, if the state variables are defined in Step 4 as

$$q_a = q_1 + q_2 \tag{10.63}$$

$$\phi_b = \phi_1 - \phi_2 \tag{10.64}$$

then Eqs. (10.41) and (10.42) become

$$\frac{dq_a}{dt} = \frac{dq_1}{dt} + \frac{dq_2}{dt} = -i_{R_1} - i_{L_1} - i_s - i_{R_3} \tag{10.65}$$

$$\frac{d\phi_b}{dt} = \frac{d\phi_1}{dt} - \frac{d\phi_2}{dt} = -v_{L_1} - v_{L_2} = v_{C_1} - v_{R_2} \tag{10.66}$$

respectively. In Step 5, the resistive link currents i_{R_1}, i_{R_3}, and the resistive tree-branch voltage V_{R_2} are solved from resistive eqs. (10.43)–(10.48) in terms of inductive link currents, capacitive tree-branch voltages, and independent sources. The results are those given in (10.55)–(10.57). By solving the inductor Eqs. (10.49), (10.50), and (10.64), inductive link current i_{L_1} is expressed as a function of state variables and independent sources:

$$i_{L_1} = \frac{1}{L_1 + L_2}\left(\phi_b - L_2 i_s\right) \tag{10.67}$$

Similarly, solving v_{C_1} from capacitor Eqs. (10.52)–(10.54), and (10.63), yields the capacitor tree-branch voltage

$$v_{C_1} = \frac{1}{C_1 + C_2}\left(q_a + C_2 v_{s_2}\right) \tag{10.68}$$

Finally, in Step 6, Eqs. (10.55)–(10.57), (10.67), and (10.68) are substituted into (10.65) and (10.66) to form the state equation in normal form:

$$\begin{bmatrix} \dfrac{dq_a}{dt} \\ \dfrac{d\phi_b}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1 + R_3}{R_1 R_3 \left(C_1 + C_2\right)} & -\dfrac{1}{L_1 + L_2} \\ \dfrac{1}{C_1 + C_2} & -\dfrac{R_2}{L_1 + L_2} \end{bmatrix} \begin{bmatrix} q_a \\ \phi_b \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1} & \dfrac{R_1 C_1 - R_3 C_2}{R_1 R_3 \left(C_1 + C_2\right)} & -\dfrac{L_1}{\left(L_1 + L_2\right)} \\ 0 & \dfrac{C_2}{\left(C_1 + C_2\right)} & -\dfrac{R_2 L_1}{\left(L_1 + L_2\right)} \end{bmatrix} \begin{bmatrix} v_{s_1} \\ v_{s_2} \\ i_s \end{bmatrix} \tag{10.69}$$

10.3 Natural Response and State Transition Matrix

In the preceding section, the state-variable description has been presented for linear time-invariant circuits. The response of the circuit depends on the solution of the state equation. The behavior of the circuit due to any arbitrary input sources can easily be obtained once the zero-input response or the natural response of the circuit is known. In order to find its natural response, the homogeneous state equation of the circuit

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{10.70}$$

is considered, where independent source term $u(t)$ has been set equal to zero. The preceding state equation is analogous to the scalar equation

$$\dot{x} = ax \tag{10.71}$$

where the solution is given by

$$x(t) = e^{at}x(0) \tag{10.72}$$

for any arbitrary initial condition $x(0)$ given at $t = 0$, or

$$x(t) = e^{a(t-t_0)}x(t_0) \tag{10.73}$$

if the initial time is specified at $t = t_0$.

It is thus reasonable to assume a solution for (10.70) of the form

$$\mathbf{x}(t) = e^{(t-t_0)\lambda}\mathbf{p} \tag{10.74}$$

where λ is a scalar constant and $\mathbf{p}$ is a constant n-vector. Substituting (10.74) into (10.70) leads to

$$\mathbf{A}\mathbf{p} = \lambda\mathbf{p} \tag{10.75}$$

Therefore, (10.74) is a solution of (10.70) precisely when p is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ. For simplicity, it is assumed that $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Because the corresponding eigenvectors denoted by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent, the general solution of (10.70) can be uniquely written as a linear combination of n distinct normal modes of the form (10.74):

$$\mathbf{x}(t) = c_1 e^{(t-t_o)\lambda_1}\mathbf{p}_1 + c_2 e^{(t-t_0)\lambda_2}\mathbf{p}_2 + \cdots + c_n e^{(t-t_0)\lambda_n}\mathbf{p}_n \tag{10.76}$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants determined by the given initial conditions. Specifically,

$$\mathbf{x}(t_0) = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + \cdots + c_n\mathbf{p}_n \tag{10.77}$$

The general solution (10.76) can also be written in the form

$$\mathbf{x}(t) = e^{(t-t_o)\mathbf{A}}\mathbf{x}(t_0) \tag{10.78}$$

where the exponential function of a matrix is defined by a power series:

$$\begin{aligned} e^{(t-t_o)\mathbf{A}} &= \mathbf{I} + (t-t_0)\mathbf{A} + \frac{(t-t_0)^2}{2!}\mathbf{A}^2 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}\mathbf{A}^k \end{aligned} \tag{10.79}$$

In fact, taking the derivative of (10.78) with respect to t yields

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \frac{d}{dt}\left[\mathbf{I} + (t-t_0)\mathbf{A} + \frac{(t-t_0)^2}{2!}\mathbf{A}^2 + \cdots\right]\mathbf{x}(t_0) \\ &= \left[\mathbf{A} + (t-t_0)\mathbf{A}^2 + \frac{(t-t_0)^2}{2!}\mathbf{A}^3 + \cdots\right]\mathbf{x}(t_0) \\ &= \mathbf{A}\left[\mathbf{I} + (t-t_0)\mathbf{A} + \frac{(t-t_0)^2}{2!}\mathbf{A}^2 + \cdots\right]\mathbf{x}(t_0) \\ &= \mathbf{A}e^{(t-t_0)\mathbf{A}}\mathbf{x}(t_0) = \mathbf{A}\mathbf{x}(t) \end{aligned} \tag{10.80}$$

Also, at $t = t_0$, (10.78) gives

$$\mathbf{x}(t_0) = \mathbf{I}\mathbf{x}(t_0) = \mathbf{x}(t_0) \tag{10.81}$$

Thus, expression (10.78) satisfies both eq. (10.70) and the initial conditions and hence is the unique solution. The matrix $e^{(t-t_0)\mathbf{A}}$, usually denoted by $\mathbf{\Phi}(t - t_0)$, is called the state transition matrix or the fundamental matrix of the circuit described by (10.70). The transition of the initial state $\mathbf{x}(t_0)$ to the state $\mathbf{x}(t)$ at any time t is thus governed by

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) \tag{10.82}$$

where

$$\mathbf{\Phi}(t - t_0) = e^{(t-t_0)\mathbf{A}} \tag{10.83}$$

is an $n \times n$ matrix with the following properties:

$$\mathbf{\Phi}(t_0 - t_0) = \mathbf{\Phi}(0) = \mathbf{I} \tag{10.84}$$

$$\mathbf{\Phi}(t + \tau) = \mathbf{\Phi}(t)\mathbf{\Phi}(\tau) \tag{10.85}$$

$$\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) = \mathbf{\Phi}(t_2 - t_0) \tag{10.86}$$

$$\mathbf{\Phi}(t_2 - t_1) = \mathbf{\Phi}^{-1}(t_1 - t_2) \tag{10.87}$$

$$\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t) \tag{10.88}$$

Once the state transition matrix is known, the solution of the state equation can be obtained from (10.82). In general, it is rather difficult to obtain a closed-form solution from the infinite series representation of the state transition matrix. The formula given by (10.79) is useful only if numerical solution by digital computer is desired. Several methods are available for finding a closed form expression for $\mathbf{\Phi}(t - t_0)$. The relationship between solution (10.76) and the state transition matrix is first established.

For simplicity, let $t_0 = 0$. According to (10.82), the first column of $\mathbf{\Phi}(t)$ is the solution of the state equation generated by the initial condition

$$\mathbf{x}(0) = \mathbf{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10.89}$$

Indeed, if (10.89) is substituted into (10.82), then

$$\mathbf{x}(t) \triangleq \mathbf{x}^{(1)}(t) = \mathbf{\Phi}(t)\mathbf{x}^{(1)}(0) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix} \tag{10.90}$$

which can be computed from (10.76) and the arbitrary constants $c_i \triangleq c_i^{(1)}$ for $i = 1, 2, \ldots, n$ are solved from (10.77). The first column of the state transition matrix is thus given by

$$\begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix} = c_1^{(1)} e^{\lambda_1 t} \mathbf{p}_1 + c_2^{(1)} e^{\lambda_2 t} \mathbf{p}_2 + \cdots + c_n^{(1)} e^{\lambda_n t} \mathbf{p}_n \tag{10.91}$$

Instead of (10.89), if

$$\mathbf{x}(0) = \mathbf{x}^{(2)}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10.92}$$

is used, the arbitrary constants $c_1, c_2, \ldots, c_n$ denoted by $c_1^{(2)}, c_2^{(2)}, \ldots c_n^{(2)}$ are solved. Then, the second column of $\mathbf{\Phi}(t)$ is given

$$\begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{bmatrix} = c_1^{(2)} e^{\lambda_1 t} \mathbf{p}_1 + c_2^{(2)} e^{\lambda_2 t} \mathbf{p}_2 + \cdots + c_n^{(2)} e^{\lambda_n t} \mathbf{p}_n \tag{10.93}$$

In a similar manner, the remaining columns of $\mathbf{\Phi}(t)$ are determined.

The closed form expression for state transition matrix can also be obtained by means of a similarity transformation of the form

$$\mathbf{AP} = \mathbf{PJ}$$

or

$$\mathbf{J} = \mathbf{P}^{-1}\mathbf{AP} \tag{10.94}$$

where $\mathbf{P}$ is a nonsingular matrix. If the eigenvalues of $\mathbf{A}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are assumed to be distinct, $\mathbf{J}$ is a diagonal matrix with eigenvalues on its main diagonal:

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{10.95}$$

and

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \tag{10.96}$$

where $\mathbf{p}_i$'s, for i = 1,2,… n, are the corresponding eigenvectors associated with the eigenvalue λ_i, for i = 1,2,…,n. Substituting (10.94) into (10.83), the state transition matrix can now be written in the closed form

$$\begin{aligned}\mathbf{\Phi}(t-t_0) &= e^{(t-t_0)\mathbf{A}} = e^{(t-t_0)\mathbf{PJP}^{-1}} \\ &= \mathbf{P}e^{(t-t_0)J}\mathbf{P}^{-1}\end{aligned} \tag{10.97}$$

where

$$e^{(t-t_o)\mathbf{J}} = \begin{bmatrix} e^{(t-t_o)\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{(t-t_o)\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(t-t_o)\lambda_n} \end{bmatrix} \tag{10.98}$$

is a diagonal matrix.

In the more general case, where the **A** matrix has repeated eigenvalues, a diagonal matrix of the form (10.95) may not exist. However, it can be shown that any square matrix **A** can be transformed by a similarity transformation to the Jordan canonical form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_l \end{bmatrix} \tag{10.99}$$

where $\mathbf{J}_i$'s, for i = 1, 2,…, l are known as Jordan blocks. Assuming that **A** has m distinct eigenvalues, λ_i, with multiplicity r_i, for $i = 1, 2, \ldots, m$, and $r_1 + r_2 + \cdots + r_m = n$. Associated with each λ_i there may exist several Jordan blocks. A Jordon block is a block diagonal matrix of order $k \times k (k \le r_i)$ with λ_i on its main diagonal, all 1's on the superdiagonal, and zeros elsewhere. In the special case when $k = 1$, the Jordan block reduces to a 1×1 scalar block with only one element $\lambda_{i.}$

In fact, the number of Jordan blocks associated with the eigenvalue λ_i is equal to the dimension of the null space of ($\lambda_i\mathbf{I} - \mathbf{A}$). For each $k \times k$ Jordan block $\mathbf{J}(k)$ associated with the eigenvalue λ_i of the form

$$\mathbf{J}(k) = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \tag{10.100}$$

the exponential function of $\mathbf{J}(k)$ takes the form

$$e^{(t-t_0)\mathbf{J}(k)} = \begin{bmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \dfrac{t^{k-2}}{(k-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} e^{(t-t_0)\lambda_i} \tag{10.101}$$

and the corresponding k columns of $\mathbf{P}$, known as the generalized eigenvectors, satisfy the equations

$$\begin{aligned}(\lambda_i\mathbf{I}-\mathbf{A})\mathbf{p}_i^{(1)} &= 0\\ (\lambda_i\mathbf{I}-\mathbf{A})\mathbf{p}_i^{(2)} &= -\mathbf{p}_i^{(1)}\\ &\vdots\\ (\lambda_i\mathbf{I}-\mathbf{A})\mathbf{p}_i^{(k)} &= -\mathbf{p}_i^{(k-1)}\end{aligned} \tag{10.102}$$

The closed form expression $\mathbf{\Phi}(t - t_0)$ for this general case now becomes

$$\mathbf{\Phi}(t-t_0) = \mathbf{P}e^{(t-t_0)J}\mathbf{P}^{-1} \tag{10.103}$$

where

$$e^{(t-t_0)\mathbf{J}} = \begin{bmatrix} e^{(t-t_0)\mathbf{J}_1} & 0 & \cdots & 0\\ 0 & e^{(t-t_0)\mathbf{J}_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{(t-t_0)\mathbf{J}_l}\end{bmatrix} \tag{10.104}$$

and each of the $e^{(t-t_0)\mathbf{J}_i}$, for $i = 1, 2, \ldots, l$, is of the form given in (10.101).

The third approach for obtaining closed form expression for the state transition matrix involves the Laplace transform technique. Taking the Laplace transform of (10.70) yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}(0) \tag{10.105}$$

where $(s\mathbf{I} - \mathbf{A})^{-1}$ is known as the resolvent matrix. The time response

$$\mathbf{x}(t) = +^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]\mathbf{x}(0) \tag{10.106}$$

is obtained by taking the inverse Laplace transform of (10.105). It is observed by comparing (10.106) to (10.82) and (10.83) with $t_0 = 0$ that

$$\mathbf{\Phi}(t) = e^{t\mathbf{A}} = \mathscr{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] \tag{10.107}$$

By way of illustration, the following example is considered. The state transition matrix is obtained by using each of the three approaches presented previously.

Example 7. Consider the parallel RLC circuit in Figure 10.7. The state equation of the circuit is obtained as

$$\begin{bmatrix}\dfrac{di_L}{dt}\\ \dfrac{dv_C}{dt}\end{bmatrix} = \begin{bmatrix}0 & \dfrac{1}{L}\\ -\dfrac{1}{C} & -\dfrac{1}{RC}\end{bmatrix}\begin{bmatrix}i_L\\ v_C\end{bmatrix} + \begin{bmatrix}0\\ \dfrac{1}{C}\end{bmatrix}i_s \tag{10.108}$$

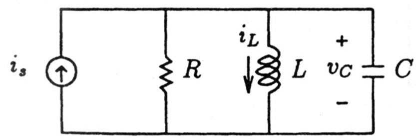

FIGURE 10.7 A parallel RLC circuit.

With $R = 2/3\ \Omega$, $L = 1\ H$, and $C = 1/2\ F$, the **A** matrix becomes

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{10.109}$$

(a) *Normal Mode Approach:* The eigenvalues and the corresponding eigenvectors of the **A** are found to be

$$\lambda_1 = -1 \quad \lambda_2 = -2 \tag{10.110}$$

and

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{10.111}$$

Therefore, the natural response of the circuit is given as a linear combination of the two distinct normal modes as

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{10.112}$$

When evaluated at $t = 0$, (10.112) becomes

$$\begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 - 2c_2 \end{bmatrix} \tag{10.113}$$

In order to find the first column of $\mathbf{\Phi}(t)$, it is assumed that

$$\begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10.114}$$

With this initial condition, the solution of (10.113) becomes

$$c_1 \triangleq c_1^{(1)} = 2 \quad \text{and} \quad c_2 \triangleq c_2^{(1)} = -1 \tag{10.115}$$

Substituting (10.115) into (10.112) results in the first column of $\mathbf{\Phi}(t)$:

$$\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} \tag{10.116}$$

Similarly, for

$$\begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10.117}$$

constants c_1 and c_2 are solved from (10.113) to give

$$c_1 \triangleq c_2^{(2)} = 1 \text{ and } c_2 \triangleq c_2^{(2)} = -1 \tag{10.118}$$

The second column of $\mathbf{\Phi}(t)$:

$$\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} \tag{10.119}$$

is obtained by substituting (10.118) into (10.112). Combining (10.116) and (10.119) yields the state transition matrix in closed form

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \tag{10.120}$$

(**b**) *Similarity Transformation Method:* The eigenvalues are distinct, so the nonsingular transformation **P** is constructed from (10.96) by the eigenvectors of **A**:

$$\mathbf{P} = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \tag{10.121}$$

with

$$\mathbf{P}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \tag{10.122}$$

Substituting λ_1, λ_2, and **P** into (10.97) and (10.98) yields the desired state transition matrix

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{P}e^{t\mathbf{J}}\mathbf{P}^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned} \tag{10.123}$$

which is in agreement with (10.120).

(**c**) *Laplace Transform Technique:* The state transition matrix can also be computed in the frequency domain from (10.107). The resolvent matrix is

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} \end{aligned} \tag{10.124}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+1} + \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

where partial-fraction expansion has been applied. Taking the inverse Laplace transform of (10.124) yields the same closed form expression as given previously in (10.120) for $\mathbf{\Phi}(t)$.

10.4 Complete Response

When independent sources are present in the circuit, the complete response depends on the initial states of the circuits as well as the input sources. It is well known that the complete response is the sum of the zero-input (or natural) response and the zero-state (or forced) response and satisfies the nonhomogeneous state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{10.125}$$

subject to the given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. Equation (10.125) is again analogous to the scalar equation

$$\dot{x}(t) = ax(t) + bu(t) \tag{10.126}$$

which has the unique solution of the form

$$x(t) = e^{(t-t_0)a}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)a}bu(\tau)\,d\tau \tag{10.127}$$

It is thus assumed that the solution to the state equation is given by

$$\begin{aligned} \mathbf{x}(t) &= e^{(t-t_0)\mathbf{A}}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \tag{10.128}$$

Indeed, one can show by direct substitution that (10.128) satisfies the state Eq. (10.125). Differentiating both sides of (10.128) with respect to t yields

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \frac{d}{dt}\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \frac{d}{dt}\int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \mathbf{A}\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \frac{d}{dt}\mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{\Phi}(t-t)\mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{A}\mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\left[\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau\right] + \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \end{aligned} \tag{10.129}$$

Also, at $t = t_0$, (10.128) becomes

$$\begin{aligned} \mathbf{x}(t_0) &= \mathbf{\Phi}(t_0-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_0} \mathbf{\Phi}(t_0-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \mathbf{I}\mathbf{x}(t_0) + \mathbf{0} = \mathbf{x}(t_0) \end{aligned} \tag{10.130}$$

The assumed solution (10.128) thus satisfies both the state Eq. (10.125) and the given initial condition. Hence, $\mathbf{x}(t)$ as given by (10.128) is the unique solution.

It is observed from (10.128) that if $\mathbf{u}(t)$ is set to zero, the solution reduces to the zero-input response or the natural response given in (10.82). On the other hand, if the original circuit is relaxed, i.e., $\mathbf{x}(t_0) = 0$, the solution represented by the convolution integral, the second term on the right-hand side of (10.128), is the forced response on the zero-state response. Thus, Eq. (10.128) verifies the fact that the complete response is the sum of the zero-input response and the zero-state response. The previous result is illustrated by means of the following example.

Example 8. Consider again the same circuit given in Example 7, where the input current source is assumed to be a unit step function applied to the circuit at $t = 0$.

The state equation of the circuit is found from (10.108) to be

$$\begin{bmatrix} \dfrac{di_L}{dt} \\ \dfrac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} i_s(t) \tag{10.131}$$

where the state transition matrix $\mathbf{\Phi}(t)$ is given in (10.120).

The zero-state response for $t > 0$ is obtained by evaluating the convolution integral indicated in (10.128):

$$\begin{aligned} \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau &= \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} d\tau \\ &= 2\int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned} \tag{10.132}$$

By adding the zero-input response represented by $\mathbf{\Phi}(t)\mathbf{x}(0)$ to (10.132), the complete response for any given initial condition $\mathbf{x}(0)$ becomes

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} + \begin{bmatrix} 1 - 2e^{-t} + 2e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} \tag{10.133}$$

for $t > 0$.

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