14

The Network Functions and Feedback[1]

Wai-Kai Chen
University of Illinois, Chicago

14.1 Blackman's Formula **14**-1
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We now study the effects of feedback on amplifier impedance and gain and obtain some useful relations among the return difference, the null return difference, and impedance functions in general.

Refer to the general feedback configuration of Figure 13.6. Let w be a transfer function. As before, to emphasize the importance of the feedback element x, we write $w = w(x)$. To be definite, let $w\,(x)$ for the time being be the current gain between the output and input ports. Then, from (13.24) we obtain

$$w(x) = \frac{I_{pq}}{I_s} = \frac{Y_2 V_{pq}}{I_s} = \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} Y_2 \tag{14.1}$$

yielding

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} \frac{Y_{uv}(0)}{Y_{rp,sq}(0)} = \frac{\hat{F}(x)}{F(x)} \tag{14.2}$$

provided that $w(0) \neq 0$. This gives a very useful formula for computing the current gain:

$$w(x) = w(0) \frac{\hat{F}(x)}{F(x)} \tag{14.3}$$

Equation (14.3) remains valid if $w(x)$ represents the transfer impedance $z_{rp,sq} = V_{pq}/I_S$ instead of the current gain.

14.1 Blackman's Formula

In particular, when $r = p$ and $s = q$, $w(x)$ represents the driving-point impedance $z_{rr,ss}(x)$ looking into the terminals r and s, and we have a somewhat different interpretation. In this case, $F(x)$ is the return difference with respect to the element x under the condition $I_s = 0$. Thus, $F(x)$ is the return difference for the situation when the port where the input impedance is defined is left open without a source and

[1]References for this chapter can be found on page **16**-17.

we write $F(x) = F(\text{input open circuited})$. Likewise, from Figure 13.6, $\hat{F}(x)$ is the return difference with respect to x for the input excitation I_s and output response V_{rs} under the condition I_s is adjusted so that V_{rs} is identically zero. Thus, $\hat{F}(x)$ is the return difference for the situation when the port where the input impedance is defined is short circuited, and we write $\hat{F}(x) = F(\text{input short-circuited})$. Consequently, the input impedance $Z(x)$ looking into a terminal pair can be conveniently expressed as

$$Z(x) = Z(0)\,\frac{F(\text{input short circuited})}{F(\text{input open circuited})} \tag{14.4}$$

This is the well-known **Blackman's formula** for computing an active impedance. The formula is extremely useful because the right-hand side can usually be determined rather easily. If x represents the controlling parameter of a controlled source in a single-loop feedback amplifier, then setting $x = 0$ opens the feedback loop and $Z(0)$ is simply a passive impedance. The return difference for x when the input port is short circuited or open circuited is relatively simple to compute because shorting out or opening a terminal pair frequently breaks the feedback loop. In addition, Blackman's formula can be used to determine the return difference by measurements. Because it involves two return differences, only one of them can be identified and the other must be known in advance. In the case of a single-loop feedback amplifier, it is usually possible to choose a terminal pair so that either the numerator or the denominator on the right-hand side of (14.4) is unity. If $F(\text{input short circuited}) = 1$, $F(\text{input open circuited})$ becomes the return difference under normal operating condition and we have

$$F(x) = \frac{Z(0)}{Z(x)} \tag{14.5}$$

On the other hand, if $F(\text{input open-circuited}) = 1$, $F(\text{input short-circuited})$ becomes the return difference under normal operating condition and we obtain

$$F(x) = \frac{Z(x)}{Z(0)} \tag{14.6}$$

Example 1. The network of Figure 14.1 is a general active RC one-port realization of a rational impedance. We use Blackman's formula to verify that its input admittance is given by

$$Y = 1 + \frac{Z_3 - Z_4}{Z_1 - Z_2} \tag{14.7}$$

Appealing to (14.4), the input admittance written as $Y = Y(x)$ can be written as

$$Y(x) = Y(0)\,\frac{F(\text{input open circuited})}{F(\text{input short circuited})} \tag{14.8}$$

where $x = 2/Z_3$. By setting x to zero, the network used to compute $Y(0)$ is shown in Figure 14.2. Its input admittance is:

$$Y(0) = \frac{Z_1 + Z_2 + Z_3 + Z_4 + 2}{Z_1 + Z_2} \tag{14.9}$$

When the input port is open-circuited, the network of Figure 14.1 degenerates to that depicted in Figure 14.3. The return difference with respect to x is:

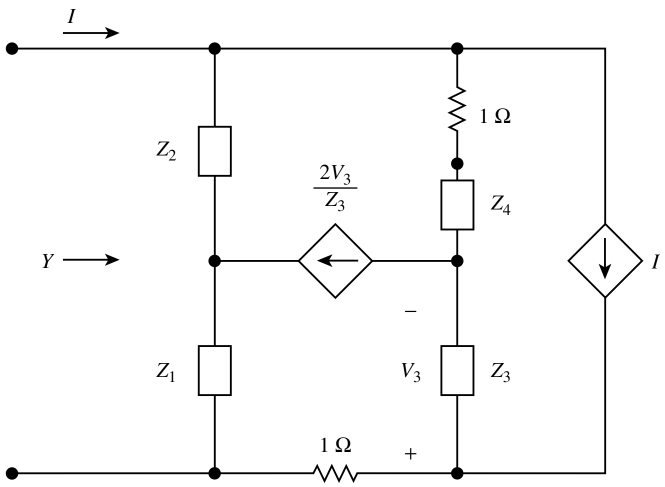

FIGURE 14.1 A general active RC one-port realization of a rational function.

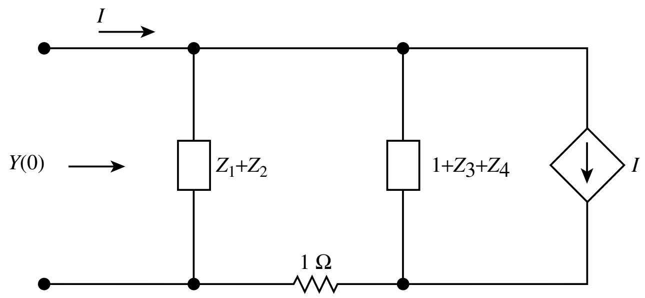

FIGURE 14.2 The network used to compute $Y(0)$.

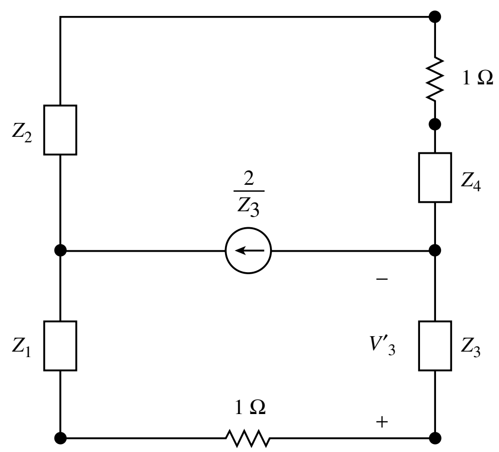

FIGURE 14.3 The network used to compute *F*(input open-circuited).

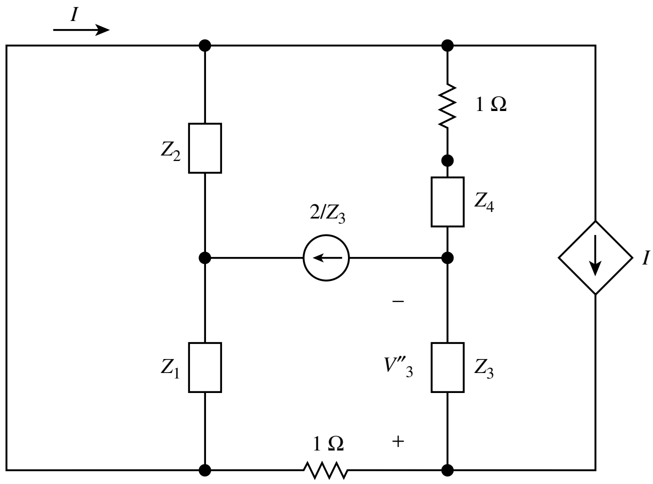

FIGURE 14.4 The network used to compute *F*(input short-circuited).

$$F(\text{input open-circuited}) = 1 - V_3' = \frac{Z_1 + Z_3 - Z_2 - Z_4}{2 + Z_1 + Z_2 + Z_3 + Z_4} \tag{14.10}$$

where the returned voltage V_3' at the controlling branch is given by

$$V_3' = \frac{2(1 + Z_2 + Z_4)}{2 + Z_1 + Z_2 + Z_3 + Z_4} \tag{14.11}$$

To compute the return difference when the input port is short circuited, we use the network of Figure 14.4 and obtain

$$F(\text{input short-circuited}) = 1 - V_3'' = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{14.12}$$

where the return voltage V_3'' at the controlling branch is found to be

$$V_3'' = \frac{2Z_2}{Z_1 + Z_2} \tag{14.13}$$

Substituting (14.9), (14.10), and (14.12) in (14.8) yields the desired result.

$$Y = 1 + \frac{Z_3 - Z_4}{Z_1 - Z_2} \tag{14.14}$$

To determine the effect of feedback on the input and output impedances, we choose the series-parallel feedback configuration of Figure 14.5. By shorting the terminals of Y_2, we interrupt the feedback loop, therefore, formula (14.5) applies and the output impedance across the load admittance Y_2 becomes

$$Z_{\text{out}}(x) = \frac{Z_{\text{out}}(0)}{F(x)} \tag{14.15}$$

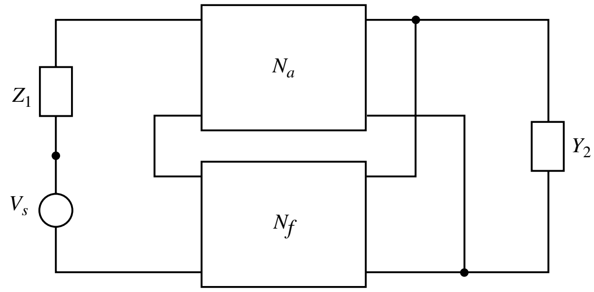

FIGURE 14.5 The series-parallel feedback configuration.

demonstrating that the impedance measured across the path of the feedback is reduced by the factor that is the normal value of the return difference with respect to the element x, where x is an arbitrary element of interest. For the input impedance of the amplifier looking into the voltage source V_s of Figure 14.5, by open circuiting or removing the voltage source V_s, we break the feedback loop. Thus, formula (14.6) applies and the input impedance becomes

$$Z_{in}(x) = F(x) Z_{in}(0) \tag{14.16}$$

meaning that the impedance measured in series lines is increased by the same factor $F(x)$. Similar conclusions can be reached for other types of configurations discussed in Chapter 12 by applying Blackman's formula.

Again, refer to the general feedback configuration of Figure 13.6 If $w(x)$ represents the voltage gain V_{pq}/V_{rs} or the transfer admittance I_{pq}/V_{rs}. Then, from (13.27) we can write

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)} \frac{Y_{rr,ss}(0)}{Y_{rr,ss}(x)} \tag{14.17}$$

The first term in the product on the right-hand side is the null return difference $\hat{F}(x)$ with respect to x for the input terminals r and s and output terminals p and q. The second term is the reciprocal of the null return difference with respect to x for the same input and output port at terminals r and s. This reciprocal can then be interpreted as the return difference with respect to x when the input port of the amplifier is short circuited. Thus, the voltage gain or the transfer admittance can be expressed as

$$w(x) = w(0) \frac{\hat{F}(x)}{F(\text{input short-circuted})} \tag{14.18}$$

Finally, if w (x)denotes the short circuit current gain I_{pq}/I_s as Y_2 approaches infinity, we obtain

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)} \frac{Y_{pp,qq}(0)}{Y_{pp,qq}(x)} \tag{14.19}$$

The second term in the product on the right-hand side is the reciprocal of the return difference with respect to x when the output port of the amplifier is short-circuited, giving a formula for the short circuit current gain as

$$w(x) = w(0) \frac{\hat{F}(x)}{F(\text{output short-circuited})} \tag{14.20}$$

Again, consider the voltage-series or series-parallel feedback amplifier of Figure 13.9 an equivalent network of which is given in Figure 13.10. The return differences $F(\tilde{\alpha}_k)$, the null return differences $\hat{F}(\tilde{\alpha}_k)$ and the voltage gain w were computed earlier in (13.45), (13.52), and (13.44), and are repeated next:

$$F(\tilde{\alpha}_1) = 93.70, \qquad F(\tilde{\alpha}_2) = 18.26 \tag{14.21a}$$

$$\hat{F}(\tilde{\alpha}_1) = 103.07 \times 10^3, \quad \hat{F}(\tilde{\alpha}_2) = 2018.70 \tag{14.21b}$$

$$w = \frac{V_{25}}{V_s} = w(\tilde{\alpha}_1) = w(\tilde{\alpha}_2) = 45.39 \tag{14.21c}$$

We apply (14.18) to calculate the voltage gain w, as follows:

$$w(\tilde{\alpha}_1) = w(0)\frac{\hat{F}(\tilde{\alpha}_1)}{F(\text{input short-circuited})} = 0.04126\,\frac{103.07 \times 10^3}{93.699} = 45.39 \tag{14.22}$$

where

$$w(0) = \left.\frac{Y_{12,55}(\tilde{\alpha}_1)}{Y_{11,55}(\tilde{\alpha}_1)}\right|_{\tilde{\alpha}_1=0} = \frac{205.24 \times 10^{-12}}{497.41 \times 10^{-11}} = 0.04126 \tag{14.23a}$$

$$F(\text{input short-circuited}) = \frac{Y_{11,55}(\tilde{\alpha}_1)}{Y_{11,55}(0)} = \frac{466.07 \times 10^{-9}}{4.9741 \times 10^{-9}} = 93.699 \tag{14.23b}$$

and

$$w(\tilde{\alpha}_2) = w(0)\frac{\hat{F}(\tilde{\alpha}_2)}{F(\text{input short-circuited})} = 0.41058\,\frac{2018.70}{18.26} = 45.39 \tag{14.24}$$

where

$$w(0) = \left.\frac{Y_{12,55}(\tilde{\alpha}_2)}{Y_{11,55}(\tilde{\alpha}_2)}\right|_{\tilde{\alpha}_2=0} = \frac{104.79 \times 10^{-10}}{255.22 \times 10^{-10}} = 0.41058 \tag{14.25a}$$

$$F(\text{input short-circuited}) = \frac{Y_{11,55}(\tilde{\alpha}_2)}{Y_{11,55}(0)} = \frac{466.07 \times 10^{-9}}{25.52 \times 10^{-9}} = 18.26 \tag{14.25b}$$

14.2 The Sensitivity Function

One of the most important effects of negative feedback is its ability to make an amplifier less sensitive to the variations of its parameters because of aging, temperature variations, or other environment changes. A useful quantitative measure for the degree of dependence of an amplifier on a particular parameter is known as the sensitivity. The **sensitivity function**, written as $\mathcal{S}(x)$, for a given transfer function with respect to an element x is defined as the ratio of the fractional change in a transfer function to the fractional change in x for the situation when all changes concerned are differentially small. Thus, if $w(x)$ is the transfer function, the sensitivity function can be written as

$$\mathcal{S}(x) = \lim_{\Delta x \to 0} \frac{\Delta w/w}{\Delta x/x} = \frac{x}{w}\frac{\partial w}{\partial x} = x\frac{\partial \ln w}{\partial x} \tag{14.26}$$

Refer to the general feedback configuration of Figure 13.6, and let $w(x)$ represent either the current gain I_{pq}/I_s or the transfer impedance V_{pq}/I_s for the time being. Then, we obtain from (13.24)

$$w(x) = Y_2 \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} \quad \text{or} \quad \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} \tag{14.27}$$

As before, we write

$$\dot{Y}_{uv}(x) = \frac{\partial Y_{uv}(x)}{\partial x} \tag{14.28a}$$

$$\dot{Y}_{rp,sq}(x) = \frac{\partial Y_{rp,sq}(x)}{\partial x} \tag{14.28b}$$

obtaining

$$Y_{uv}(x) = Y_{uv}(0) + x\dot{Y}_{uv}(x) \tag{14.29a}$$

$$Y_{rp,sq}(x) = Y_{rp,sq}(0) + x\dot{Y}_{rp,sq}(x) \tag{14.29b}$$

Substituting (14.27) in (14.26), in conjunction with (14.29), yields

$$\begin{aligned}\mathcal{S}(x) &= x\frac{\dot{Y}_{rp,sq}(x)}{Y_{rp,sq}(x)} - x\frac{\dot{Y}_{uv}(x)}{Y_{uv}(x)} = \frac{Y_{rp,sq}(x) - Y_{rp,sq}(0)}{Y_{rp,sq}(x)} - \frac{Y_{uv}(x) - Y_{uv}(0)}{Y_{uv}(x)} \\ &= \frac{Y_{uv}(0)}{Y_{uv}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(x)} - \frac{1}{\hat{F}(x)}\end{aligned} \tag{14.30}$$

Combining this with (14.3), we obtain

$$\mathcal{S}(x) = \frac{1}{F(x)}\left[1 - \frac{w(0)}{w(x)}\right] \tag{14.31}$$

Observe that if $w(0) = 0$, (14.31) becomes

$$\mathcal{S}(x) = \frac{1}{F(x)} \tag{14.32}$$

meaning that sensitivity is equal to the reciprocal of the return difference. For the ideal feedback model, the feedback path is unilateral. Hence, $w(0) = 0$ and

$$\mathcal{S} = \frac{1}{F} = \frac{1}{1+T} = \frac{1}{1-\mu\beta} \tag{14.33}$$

For a practical amplifier, $w(0)$ is usually very much smaller than $w(x)$ in the passband, and $F \approx 1/\mathcal{S}$ may be used as a good estimate of the reciprocal of the sensitivity in the same frequency band. A single-loop feedback amplifier composed of a cascade of common-emitter stages with a passive network providing the desired feedback fulfills this requirements. If in such a structure any one of the transistors fails, the forward transmission is nearly zero and $w(0)$ is practically zero. Our conclusion is that if the failure of any element will interrupt the transmission through the amplifier as a whole to nearly zero, the sensitivity is approximately equal to the reciprocal of the return difference with respect to that element.

In the case of driving-point impedance, $w(0)$ is not usually smaller than $w(x)$, and the reciprocity relation is not generally valid.

Now assume that $w(x)$ represents the voltage gain. Substituting (14.27) in (14.26) results in

$$\begin{aligned}\mathcal{S}(x) &= x\frac{\dot{Y}_{rp,sq}(x)}{Y_{rp,sq}(x)} - x\frac{\dot{Y}_{rr,ss}(x)}{Y_{rr,ss}(x)} = \frac{Y_{rp,sq}(x)-Y_{rp,sq}(0)}{Y_{rp,sq}(x)} - \frac{Y_{rr,ss}(x)-Y_{rr,ss}(0)}{Y_{rr,ss}(x)} \\ &= \frac{Y_{rr,ss}(0)}{Y_{rr,ss}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(\text{input short-circuited})} - \frac{1}{\hat{F}(x)}\end{aligned} \tag{14.34}$$

Combining this with (14.18) gives

$$\mathcal{S}(x) = \frac{1}{F(\text{input short-circuited})}\left[1 - \frac{w(0)}{w(x)}\right] \tag{14.35}$$

Finally, if $w(x)$ denotes the short circuit current gain I_{pq}/I_s as Y_2 approaches infinity, the sensitivity function can be written as

$$\mathcal{S}(x) = \frac{Y_{pp,qq}(0)}{Y_{pp,qq}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(\text{output short-circuited})} - \frac{1}{\hat{F}(x)} \tag{14.36}$$

which when combined with (14.20) yields

$$\mathcal{S}(x) = \frac{1}{F(\text{output short-circuited})}\left[1 - \frac{w(0)}{w(x)}\right] \tag{14.37}$$

We remark that formulas (14.31), (14.35), and (14.39) are quite similar. If the return difference $F(x)$ is interpreted properly, they can all be represented by the single relation (14.31). As before, if $w(0) = 0$, the sensitivity for the voltage gain function is equal to the reciprocal of the return difference under the situation that the input port of the amplifier is short-circuited, whereas the sensitivity for the short circuit current gain is the reciprocal of the return difference when the output port is short-circuited.

Example 2. The network of Figure 14.6 is a common-emitter transistor amplifier. After removing the biasing circuit and using the common-emitter hybrid model for the transistor at low frequencies, an equivalent network of the amplifier is presented in Figure 14.7 with

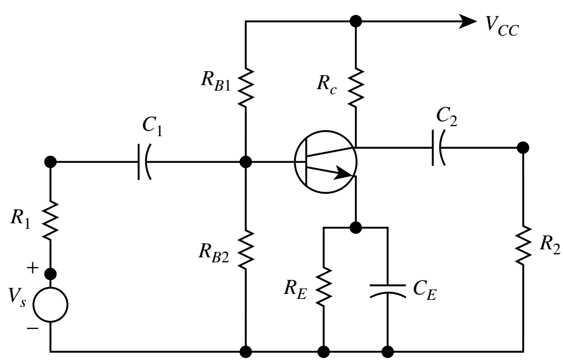

FIGURE 14.6 A common-emitter transistor feedback amplifier.

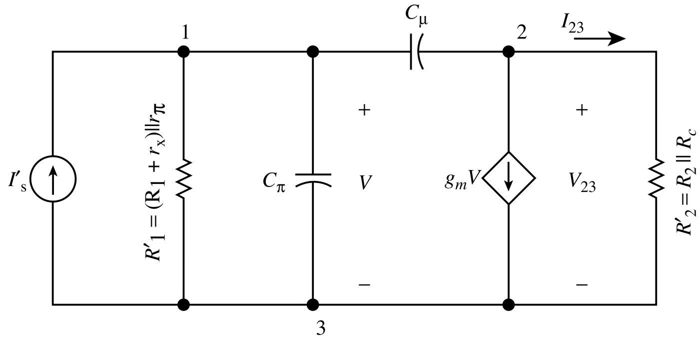

FIGURE 14.7 An equivalent network of the feedback amplifier of Figure 14.6.

$$I'_s = \frac{V_s}{R_1 + r_x} \tag{14.38a}$$

$$G'_1 = \frac{1}{R'_1} = \frac{1}{R_1 + r_x} + \frac{1}{r_\pi} \tag{14.38b}$$

$$G'_2 = \frac{1}{R'_2} = \frac{1}{R_2} + \frac{1}{R_c} \tag{14.38c}$$

The indefinite admittance matrix of the amplifier is:

$$\mathbf{Y} = \begin{bmatrix} G'_1 + sC_\pi + sC_\mu & -sC_\mu & -G'_1 - sC_\pi \\ g_m - sC_\mu & G'_2 + sC_\mu & -G'_2 - g_m \\ -G'_1 - sC_\pi - g_m & -G'_2 & G'_1 + G'_2 + sC_\pi + g_m \end{bmatrix} \tag{14.39}$$

Assume that the controlling parameter g_m is the element of interest. The return difference and the null return difference with respect to g_m in Figure 14.7 with I'_s as the input port and R'_2, as the output port, are:

$$F(g_m) = \frac{Y_{33}(g_m)}{Y_{33}(0)} = \frac{(G'_1 + sC_\pi)(G'_2 + sC_\mu) + sC_\mu(G'_2 + g_m)}{(G'_1 + sC_\pi)(G'_2 + sC_\mu) + sC_\mu G'_2} \tag{14.40}$$

$$\hat{F}(g_m) = \frac{Y_{12,33}(g_m)}{Y_{12,33}(0)} = \frac{sC_\mu - g_m}{sC_\mu} = 1 - \frac{g_m}{sC_\mu} \tag{14.41}$$

The current gain I_{23}/I'_s as defined in Figure 14.7, is computed as

$$w(g_m) = \frac{Y_{12,33}(g_m)}{R'_2 Y_{33}(g_m)} = \frac{sC_\mu - g_m}{R'_2\left[(G'_1 + sC_\pi)(G'_2 + sC_\mu) + sC_\mu(G'_2 + g_m)\right]} \tag{14.42}$$

Substituting these in (14.30) or (14.31) gives

$$\mathcal{S}(g_m) = -\frac{g_m(G'_1 + sC_\pi + sC_\mu)(G'_2 + sC_\mu)}{(sC_\mu - g_m)\left[(G'_1 + sC_\pi)(G'_2 + sC_\mu) + sC_\mu(G'_2 + g_m)\right]} \tag{14.43}$$

Finally, we compute the sensitivity for the driving-point impedance facing the current source I_s'. From (14.31), we obtain

$$\mathcal{S}(g_m) = \frac{1}{F(g_m)}\left[1 - \frac{Z(0)}{Z(g_m)}\right] = -\frac{sC_\mu g_m}{(G_1' + sC_\pi)(G_2' + sC_\mu) + sC_\mu(G_2' + g_m)} \tag{14.44}$$

where

$$Z(g_m) = \frac{Y_{11,33}(g_m)}{Y_{33}(g_m)} = \frac{G_2' + sC_\mu}{(G_1' + sC_\pi)(G_2' + sC_\mu) + sC_\mu(G_2' + g_m)} \tag{14.45}$$