16
Multiple-Loop Feedback Amplifiers

16.1 Multiple-Loop Feedback Amplifier Theory **16**-1
16.2 The Return Difference Matrix.. **16**-5
16.3 The Null Return Difference Matrix **16**-6
16.4 The Transfer-Function Matrix and Feedback **16**-8
16.5 The Sensitivity Matrix **16**-11
16.6 Multiparameter Sensitivity **16**-15

Wai-Kai Chen
University of Illinois, Chicago

So far, we have studied the single-loop feedback amplifiers. The concept of feedback was introduced in terms of return difference. We found that return difference is the difference between the unit applied signal and the returned signal. The returned signal has the same physical meaning as the loop transmission in the ideal feedback mode. It plays an important role in the study of amplifier stability, its sensitivity to the variations of the parameters, and the determination of its transfer and driving point impedances. The fact that return difference can be measured experimentally for many practical amplifiers indicates that we can include all the parasitic effects in the stability study, and that stability problem can be reduced to a Nyquist plot.

In this section, we study amplifiers that contain a multiplicity of inputs, outputs, and feedback loops. They are referred to as the **multiple-loop feedback amplifiers**. As might be expected, the notion of return difference with respect to an element is no longer applicable, because we are dealing with a group of elements. For this, we generalize the concept of return difference for a controlled source to the notion of return difference matrix for a multiplicity of controlled sources. For measurement situations, we introduce the null return difference matrix and discuss its physical significance. We demonstrate that the determinant of the overall transfer function matrix can be expressed explicity in terms of the determinants of the return difference and the null return difference matrices, thereby allowing us to generalize Blackman's formula for the input impedance.

16.1 Multiple-Loop Feedback Amplifier Theory

The general configuration of a multiple-input, multiple-output, and multiple-loop feedback amplifier is presented in Figure 16.1, in which the input, output, and feedback variables may be either currents or voltages. For the specific arrangement of Figure 16.1, the input and output variables are represented by an n-dimensional vector $\mathbf{u}$ and an m-dimensional vector $\mathbf{y}$ as

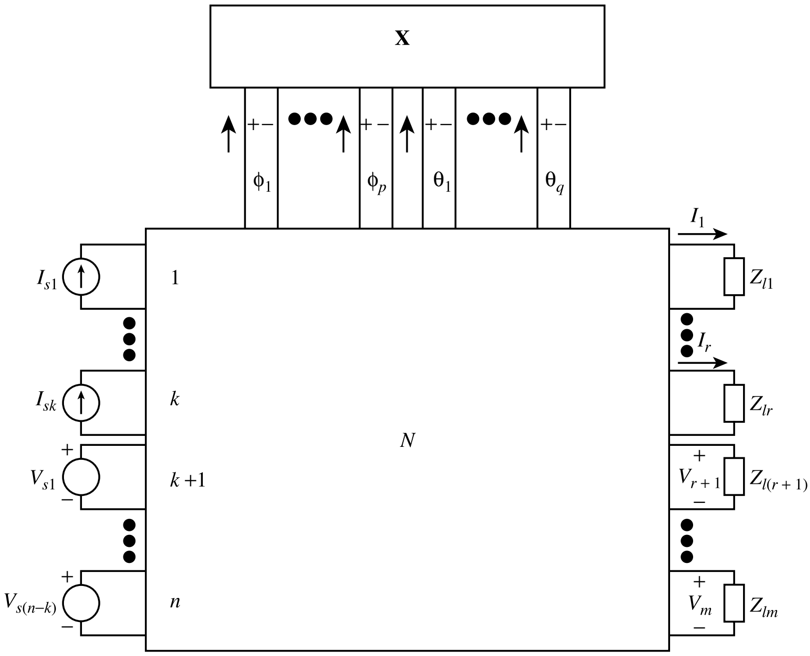

FIGURE 16.1 The general configuration of a multiple-input, multiple-output, and multiple-loop feedback amplifier.

$$\mathbf{u}(s) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \\ u_{k+1} \\ u_{k+2} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} I_{s1} \\ I_{s2} \\ \vdots \\ I_{sk} \\ V_{s1} \\ V_{s2} \\ \vdots \\ V_{s(n-k)} \end{bmatrix}, \quad \mathbf{y}(s) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \\ y_{r+1} \\ y_{r+2} \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_r \\ V_{r+1} \\ V_{r+2} \\ \vdots \\ V_m \end{bmatrix} \tag{16.1}$$

respectively. The elements of interest can be represented by a rectangular matrix **X** of order $q \times p$ relating the controlled and controlling variables by the matrix equation

$$\mathbf{\Theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{q1} & x_{q2} & \cdots & x_{qp} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \mathbf{X\Phi} \tag{16.2}$$

where the p-dimensional vector $\mathbf{\Phi}$ is called the **controlling vector**, and the q-dimensional vector $\mathbf{\Theta}$ is the **controlled vector**. The controlled variables θ_k and the controlling variables Φ_k can either be currents or voltages. The matrix **X** can represent either a transfer-function matrix or a driving-point function matrix.

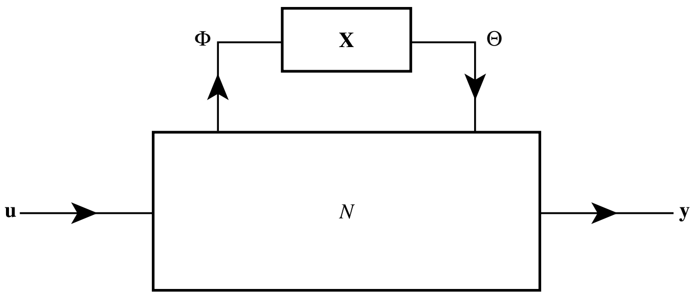

FIGURE 16.2 The block diagram of the general feedback configuration of Figure 16.1.

If **X** represents a driving-point function matrix, the vectors **Θ** and **Φ** are of the same dimension ($q = p$) and their components are the currents and voltages of a p-port network.

The general configuration of Figure 16.1 can be represented equivalently by the block diagram of Figure 16.2 in which N is a $(p + q + m + n)$-port network and the elements of interest are exhibited explicitly by the block **X**. For the $(p + q + m + n)$-port network N, the vectors **u** and are **Θ** are its inputs, and the vectors **Φ** and **y** its outputs. Since N is linear, the input and output vectors are related by the matrix equations

$$\mathbf{\Phi} = \mathbf{A\Theta} + \mathbf{Bu} \tag{16.3a}$$

$$\mathbf{y} = \mathbf{C\Theta} + \mathbf{Du} \tag{16.3b}$$

where **A**, **B**, **C**, and **D** are transfer-function matrices of orders $p \times q$, $p \times n$, $m \times q$, and $m \times n$, respectively. The vectors **Θ** and **Φ** are not independent and are related by

$$\mathbf{\Theta} = \mathbf{X\Phi} \tag{16.3c}$$

The relationships among the above three linear matrix equations can also be represented by a matrix signal-flow graph as shown in Figure 16.3 know as the **fundamental matrix feedback-flow graph**. The overall closed-loop **transfer-function matrix** of the multiple-loop feedback amplifier is defined by the equation

$$\mathbf{y} = \mathbf{W}(\mathbf{X})\mathbf{u} \tag{16.4}$$

where $\mathbf{W}(\mathbf{X})$ is of order $m \times n$. As before, to emphasize the importance of **X**, the matrix **W** is written as $\mathbf{W}(\mathbf{X})$ for the present discussion, even though it is also a function of the complex-frequency variable s. Combining the previous matrix equations, the transfer-function matrix is:

$$\mathbf{W}(\mathbf{X}) = \mathbf{D} + \mathbf{CX}\left(\mathbf{1}_p - \mathbf{AX}\right)^{-1}\mathbf{B} \tag{16.5a}$$

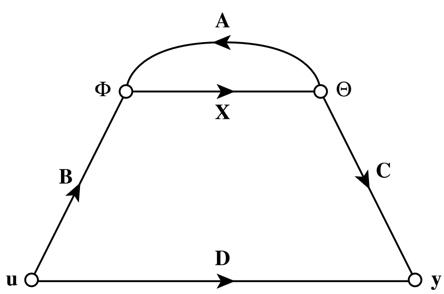

FIGURE 16.3 The fundamental matrix feedback-flow graph.

or

$$\mathbf{W}(\mathbf{X}) = \mathbf{D} + \mathbf{C}\left(\mathbf{1}_q - \mathbf{X}\mathbf{A}\right)^{-1}\mathbf{X}\mathbf{B} \tag{16.5b}$$

where $\mathbf{1}_p$ denotes the identity matrix of order p. Clearly, we have

$$\mathbf{W}(\mathbf{0}) = \mathbf{D} \tag{16.6}$$

In particular, when $\mathbf{X}$ is square and nonsingular, (16.5) can be written as

$$\mathbf{W}(\mathbf{X}) = \mathbf{D} + \mathbf{C}\left(\mathbf{X}^{-1} - \mathbf{A}\right)^{-1}\mathbf{B} \tag{16.7}$$

Example 3. Consider the voltage-series feedback amplifier of Figure 13.9. An equivalent network is shown in Figure 16.4 in which we have assumed that the two transistors are identical with $h_{ie} = 1.1$ kΩ, $h_{fe} = 50$, $h_{re} = h_{oe} = 0$. Let the controlling parameters of the two controlled sources be the elements of interest. Then we have

$$\mathbf{\Theta} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} = 10^{-4}\begin{bmatrix} 455 & 0 \\ 0 & 455 \end{bmatrix}\begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix} = \mathbf{X\Phi} \tag{16.8}$$

Assume that the output voltage V_{25} and input current I_{51} are the output variables. Then the seven-port network N defined by the variables V_{13}, V_{45}, V_{25}, I_{51}, I_a, I_b, and V_s can be characterized by the matrix equations

$$\begin{aligned}\mathbf{\Phi} = \begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix} &= \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + \begin{bmatrix} 0.91748 \\ 0 \end{bmatrix}\left[V_s\right] \\ &= \mathbf{A\Theta} + \mathbf{Bu}\end{aligned} \tag{16.9a}$$

$$\begin{aligned}\mathbf{y} = \begin{bmatrix} V_{25} \\ I_{51} \end{bmatrix} &= \begin{bmatrix} 45.391 & -2372.32 \\ -0.08252 & 0.04126 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + \begin{bmatrix} 0.041260 \\ 0.000862 \end{bmatrix}\left[V_s\right] \\ &= \mathbf{C\Theta} + \mathbf{Du}\end{aligned} \tag{16.9b}$$

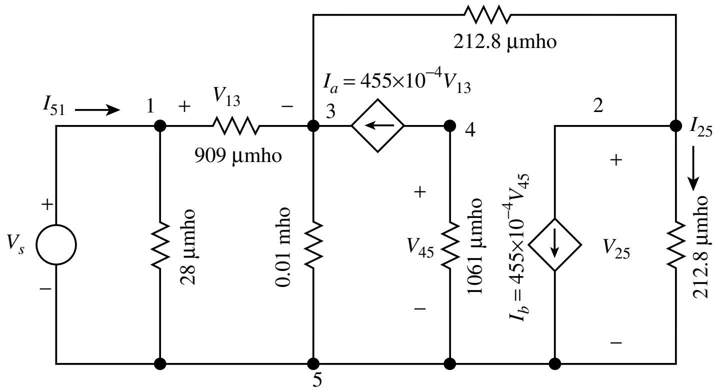

FIGURE 16.4 An equivalent network of the voltage-series feedback amplifier of Figure 13.9.

According to (16.4), the transfer-function matrix of the amplifier is defined by the matrix equation

$$\mathbf{y} = \begin{bmatrix} V_{25} \\ I_{51} \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} [V_s] = \mathbf{W}(\mathbf{X})\mathbf{u} \tag{16.10}$$

Because **X** is square and nonsingular, we can use (16.7) to calculate **W**(**X**):

$$\mathbf{W}(\mathbf{X}) = \mathbf{D} + \mathbf{C}(\mathbf{X}^{-1} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 45.387 \\ 0.369 \times 10^{-4} \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} \tag{16.11}$$

where

$$(\mathbf{X}^{-1} - \mathbf{A})^{-1} = 10^{-4} \begin{bmatrix} 4.856 & 10.029 \\ -208.245 & 24.914 \end{bmatrix} \tag{16.12}$$

obtaining the closed-loop voltage gain w_{11} and input impedance Z_{in} facing the voltage source V_s as

$$w_{11} = \frac{V_{25}}{V_s} = 45.387, \quad Z_{\text{in}} = \frac{V_s}{I_{51}} = \frac{1}{w_{21}} = 27.1\ \mathbf{k\Omega} \tag{16.13}$$

16.2 The Return Different Matrix

In this section, we extend the concept of return difference with respect to an element to the notion of return difference matrix with respect to a group of elements.

In the fundamental matrix feedback-flow graph of Figure 16.3, suppose that we break the input of the branch with transmittance **X**, set the input excitation vector **u** to zero, and apply a signal p-vector **g** to the right of the breaking mark, as depicted in Figure 16.5. Then the returned signal p-vector **h** to the left of the breaking mark is found to be

$$\mathbf{h} = \mathbf{AXg} \tag{16.14}$$

The square matrix **AX** is called the **loop-transmission matrix** and its negative is referred to as the **return ratio matrix** denoted by

$$\mathbf{T}(\mathbf{X}) = -\mathbf{AX} \tag{16.15}$$

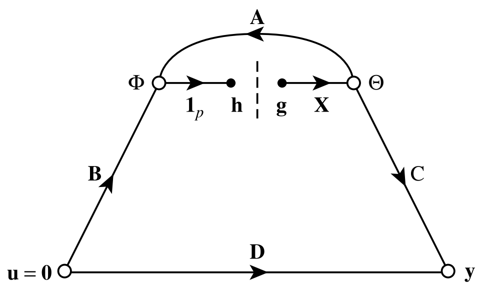

FIGURE 16.5 The physical interpretation of the loop-transmission matrix.

The difference between the applied signal vector **g** and the returned signal vector **h** is given by

$$\mathbf{g}-\mathbf{h}=\left(\mathbf{1}_p-\mathbf{AX}\right)\mathbf{g} \tag{16.16}$$

The square matrix $\mathbf{1}_p - \mathbf{AX}$ relating the applied signal vector **g** to the difference of the applied signal vector **g** and the returned signal vector **h** is called the **return difference matrix** with respect to **X** and is denoted by

$$\mathbf{F}\left(\mathbf{X}\right)=\mathbf{1}_p-\mathbf{AX} \tag{16.17}$$

Combining this with (16.15) gives

$$\mathbf{F}\left(\mathbf{X}\right)=\mathbf{1}_p+\mathbf{T}\left(\mathbf{X}\right) \tag{16.18}$$

For the voltage-series feedback amplifier of Figure 16.4, let the controlling parameters of the two controlled current sources be the elements of interest. Then the return ratio matrix is found from (16.8) and (16.9a)

$$\begin{aligned}\mathbf{T}\left(\mathbf{X}\right)&=-\mathbf{AX}=-\begin{bmatrix}-90.782 & 45.391\\ -942.507 & 0\end{bmatrix}\begin{bmatrix}455\times10^{-4} & 0\\ 0 & 455\times10^{-4}\end{bmatrix}\\ &=\begin{bmatrix}4.131 & -2.065\\ 42.884 & 0\end{bmatrix}\end{aligned} \tag{16.19}$$

obtaining the return difference matrix as

$$\mathbf{F}\left(\mathbf{X}\right)=\mathbf{1}_2+\mathbf{T}\left(\mathbf{X}\right)=\begin{bmatrix}5.131 & -2.065\\ 42.884 & 1\end{bmatrix} \tag{16.20}$$

16.3 The Null Return Difference Matrix

A direct extension of the null return difference for the single-loop feedback amplifier is the null return difference matrix for the multiple-loop feedback networks.

Refer again to the fundamental matrix feedback-flow graph of Figure 16.3. As before, we break the branch with transmittance **X** and apply a signal p-vector **g** to the right of the breaking mark, as illustrated in Figure 16.6. We then adjust the input excitation n-vector **u** so that the total output m-vector **y** resulting from the inputs **g** and **u** is zero. From Figure 16.6, the desired input excitation **u** is found:

$$\mathbf{Du}+\mathbf{CXg}=\mathbf{0} \tag{16.21}$$

or

$$\mathbf{u}=-\mathbf{D}^{-1}\mathbf{CXg} \tag{16.22}$$

provided that the matrix **D** is square and nonsingular. This requires that the output **y** be of the same dimension as the input **u** or $m = n$. Physically, this requirement is reasonable because the effects at the output caused by **g** can be neutralized by a unique input excitation **u** only when **u** and **y** are of the same dimension. With these inputs **u** and **g**, the returned signal **h** to the left of the breaking mark in Figure 16.6 is computed as

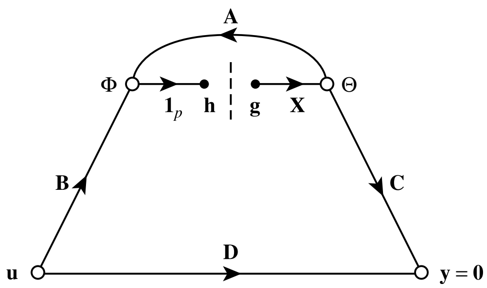

FIGURE 16.6 The physical interpretation of the null return difference matrix.

$$\mathbf{h} = \mathbf{Bu} + \mathbf{AXg} = \left(-\mathbf{BD}^{-1}\mathbf{CX} + \mathbf{AX}\right)\mathbf{g} \tag{16.23}$$

obtaining

$$\mathbf{g} - \mathbf{h} = \left(\mathbf{1}_p - \mathbf{AX} + \mathbf{BD}^{-1}\mathbf{CX}\right)\mathbf{g} \tag{16.24}$$

The square matrix

$$\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{1}_p + \hat{\mathbf{T}}(\mathbf{X}) = \mathbf{1}_p - \mathbf{AX} + \mathbf{BD}^{-1}\mathbf{CX} = \mathbf{1}_p - \hat{\mathbf{A}}\mathbf{X} \tag{16.25}$$

relating the input signal vector **g** to the difference of the input signal vector **g**, and the returned signal vector **h** is called the **null return difference matrix** with respect to **X**, where

$$\hat{\mathbf{T}}(\mathbf{X}) = -\mathbf{AX} + \mathbf{BD}^{-1}\mathbf{CX} = -\hat{\mathbf{A}}\mathbf{X} \tag{16.26a}$$

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{BD}^{-1}\mathbf{C} \tag{16.26b}$$

The square matrix $\hat{\mathbf{T}}(\mathbf{X})$ is known as the *null return ratio matrix.*

Example 4. Consider again the voltage-series feedback amplifier of Figure 13.9, an equivalent network of which is illustrated in Figure 16.4. Assume that the voltage V_{25} is the output variable. Then from (16.9)

$$\begin{aligned}\boldsymbol{\Phi} = \begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix} &= \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + \begin{bmatrix} 0.91748 \\ 0 \end{bmatrix}\begin{bmatrix} V_s \end{bmatrix} \\ &= \mathbf{A}\boldsymbol{\Theta} + \mathbf{B}u \end{aligned} \tag{16.27a}$$

$$\begin{aligned} y = \begin{bmatrix} V_{25} \end{bmatrix} &= \begin{bmatrix} 45.391 & -2372.32 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + \begin{bmatrix} 0.04126 \end{bmatrix}\begin{bmatrix} V_s \end{bmatrix} \\ &= \mathbf{C}\boldsymbol{\Theta} + \mathbf{D}u \end{aligned} \tag{16.27b}$$

Substituting the coefficient matrices in (16.26b), we obtain

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{BD}^{-1}\mathbf{C} = \begin{bmatrix} -1100.12 & 52{,}797.6 \\ -942.507 & 0 \end{bmatrix} \tag{16.28}$$

giving the null return difference matrix with respect to **X** as

$$\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{1}_2 - \hat{\mathbf{A}}\mathbf{X} = \begin{bmatrix} 51.055 & -2402.29 \\ 42.884 & 1 \end{bmatrix} \tag{16.29}$$

Suppose that the input current I_{51} is chosen as the output variable. Then, from (16.9b) we have

$$y = [I_{51}] = [-0.08252 \quad 0.04126]\begin{bmatrix} I_a \\ I_b \end{bmatrix} + [0.000862][V_s] = \mathbf{C\Theta} + Du \tag{16.30}$$

The corresponding null return difference matrix becomes

$$\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{1}_2 - \hat{\mathbf{A}}\mathbf{X} = \begin{bmatrix} 1.13426 & -0.06713 \\ 42.8841 & 1 \end{bmatrix} \tag{16.31}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} -2.95085 & 1.47543 \\ -942.507 & 0 \end{bmatrix} \tag{16.32}$$

16.4 The Transfer-Function Matrix and Feedback

In this section, we show the effect of feedback on the transfer-function matrix $\mathbf{W}(\mathbf{X})$. Specifically, we express det $\mathbf{W}(\mathbf{X})$ in terms of the det $\mathbf{X}(\mathbf{0})$ and the determinants of the return difference and null return difference matrices, thereby generalizing Blackman's impedance formula for a single input to a multiplicity of inputs.

Before we proceed to develop the desired relation, we state the following determinant identity for two arbitrary matrices **M** and **N** of order $m \times n$ and $n \times m$:

$$\det(\mathbf{1}_m + \mathbf{MN}) = \det(\mathbf{1}_n + \mathbf{NM}) \tag{16.33}$$

a proof of which may be found in [5, 6]. Using this, we next establish the following generalization of Blackman's formula for input impedance.

Theorem 1. *In a multiple-loop feedback amplifier, if* $\mathbf{W}(\mathbf{0}) = \mathbf{D}$ *is nonsingular, then the determinant of the transfer-function matrix* $\mathbf{W}(\mathbf{X})$ *is related to the determinants of the return difference matrix* $\mathbf{F}(\mathbf{X})$ *and the null return difference matrix* $\hat{\mathbf{F}}(\mathbf{X})$ *by*

$$\det \mathbf{W}(\mathbf{X}) = \det \mathbf{W}(\mathbf{0}) \frac{\det \hat{\mathbf{F}}(\mathbf{X})}{\det \mathbf{F}(\mathbf{X})} \tag{16.34}$$

PROOF: From (16.5a), we obtain

$$\mathbf{W}(\mathbf{X}) = \mathbf{D}\left[\mathbf{1}_n + \mathbf{D}^{-1}\mathbf{CX}(\mathbf{1}_p - \mathbf{AX})^{-1}\mathbf{B}\right] \tag{16.35}$$

yielding

$$
\begin{aligned}
\det \mathbf{W}(\mathbf{X}) &= [\det \mathbf{W}(\mathbf{0})] \det\left[\mathbf{1}_n + \mathbf{D}^{-1}\mathbf{C}\mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\mathbf{B}\right] \\
&= [\det \mathbf{W}(\mathbf{0})] \det\left[\mathbf{1}_p + \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\right] \\
&= [\det \mathbf{W}(\mathbf{0})] \det\left[\mathbf{1}_p - \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\mathbf{X}\right](\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1} \\
&= \frac{\det \mathbf{W}(\mathbf{0}) \det \hat{\mathbf{F}}(\mathbf{X})}{\det \mathbf{F}(\mathbf{X})}
\end{aligned}
\tag{16.36}
$$

The second line follows directly from (16.33). This completes the proof of the theorem.

As indicated in (14.4), the input impedance $Z(x)$ looking into a terminal pair can be conveniently expressed as

$$
Z(x) = Z(0)\frac{F(\text{input short-ciruited})}{F(\text{input open-circuited})} \tag{16.37}
$$

A similar expression can be derived from (16.34) if $\mathbf{W}(\mathbf{X})$ denotes the impedance matrix of an n-port network of Figure 16.1. In this case, $\mathbf{F}(\mathbf{X})$ is the return difference matrix with respect to $\mathbf{X}$ for the situation when the n ports where the impedance matrix are defined are left open without any sources, and we write $\mathbf{F}(\mathbf{X}) = \mathbf{F}(\text{input open-circuited})$. Likewise, $\hat{\mathbf{F}}(\mathbf{X})$ is the return difference matrix with respect to $\mathbf{X}$ for the input port-current vector $\mathbf{I}_s$ and the output port-voltage vector $\mathbf{V}$ under the condition that $\mathbf{I}_s$ is adjusted so that the port-voltage vector $\mathbf{V}$ is identically zero. In other words, $\hat{\mathbf{F}}(\mathbf{X})$ is the return difference matrix for the situation when the n ports, where the impedance matrix is defined, are short-circuited, and we write $\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{F}(\text{input short-circuited})$. Consequently, the determinant of the impedance matrix $\mathbf{Z}(\mathbf{X})$ of an n-port network can be expressed from (16.34) as

$$
\det \mathbf{Z}(\mathbf{X}) = \det \mathbf{Z}(\mathbf{0})\frac{\det \mathbf{F}(\text{input short-circuited})}{\det \mathbf{F}(\text{input open-circuited})} \tag{16.38}
$$

Example 5. Refer again to the voltage-series feedback amplifier of Figure 13.9, an equivalent network of which is illustrated in Figure 16.4. As computed in (16.20), the return difference matrix with respect to the two controlling parameters is given by

$$
\mathbf{F}(\mathbf{X}) = \mathbf{1}_2 + \mathbf{T}(\mathbf{X}) = \begin{bmatrix} 5.131 & -2.065 \\ 42.884 & 1 \end{bmatrix} \tag{16.39}
$$

the determinant of which is:

$$
\det \mathbf{F}(\mathbf{X}) = 93.68646 \tag{16.40}
$$

If V_{25} of Figure 16.4 is chosen as the output and V_s as the input, the null return difference matrix is, from (16.29),

$$
\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{1}_2 - \hat{\mathbf{A}}\mathbf{X} = \begin{bmatrix} 51.055 & -2402.29 \\ 42.884 & 1 \end{bmatrix} \tag{16.41}
$$

the determinant of which is:

$$\det \hat{\mathbf{F}}(\mathbf{X}) = 103{,}071 \tag{16.42}$$

By appealing to (16.34), the feedback amplifier voltage gain V_{25}/V_s can be written as

$$w(\mathbf{X}) = \frac{V_{25}}{V_s} = w(\mathbf{0}) \frac{\det \hat{\mathbf{F}}(\mathbf{X})}{\det \mathbf{F}(\mathbf{X})} = 0.04126 \frac{103{,}071}{93.68646} = 45.39 \tag{16.43}$$

confirming (13.44), where $w(\mathbf{0}) = 0.04126$, as given in (16.27b).

Suppose, instead, that the input current I_{51} is chosen as the output and V_s as the input. Then, from (16.31), the null return difference matrix becomes

$$\hat{\mathbf{F}}(\mathbf{X}) = \mathbf{1}_2 - \hat{\mathbf{A}}(\mathbf{X}) = \begin{bmatrix} 1.13426 & -0.06713 \\ 42.8841 & 1 \end{bmatrix} \tag{16.44}$$

the determinant of which is:

$$\det \hat{\mathbf{F}}(\mathbf{X}) = 4.01307 \tag{16.45}$$

By applying (16.34), the amplifier input admittance is obtained as

$$\begin{aligned} w(\mathbf{X}) &= \frac{I_{51}}{V_s} = w(\mathbf{0}) \frac{\det \hat{\mathbf{F}}(\mathbf{X})}{\det \mathbf{F}(\mathbf{X})} \\ &= 8.62 \times 10^{-4} \frac{4.01307}{93.68646} = 36.92\ \mu\text{mho} \end{aligned} \tag{16.46}$$

or 27.1 kΩ, confirming (16.13), where $w(\mathbf{0}) = 862\ \mu$mho is found from (16.30).

Another useful application of the generalized Blackman's formula (16.38) is that it provides the basis of a procedure for the indirect measurement of return difference. Refer to the general feedback network of Figure 16.2. Suppose that we wish to measure the return difference $F(y_{21})$ with respect to the forward short circuit transfer admittance y_{21} of a two-port device characterized by its y parameters y_{ij}. Choose the two controlling parameters y_{21} and y_{12} to be the elements of interest. Then, from Figure 15.2 we obtain

$$\mathbf{\Theta} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} y_{21} & 0 \\ 0 & y_{12} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{X\Phi} \tag{16.47}$$

where I_a and I_b are the currents of the voltage-controlled current sources. By appealing to (16.38), the impedance looking into terminals a and b of Figure 15.2 can be written as

$$z_{aa,bb}(y_{12}, y_{21}) = z_{aa,bb}(0,0) \frac{\det \mathbf{F}(\text{input short-circuited})}{\det \mathbf{F}(\text{input open-circuited})} \tag{16.48}$$

When the input terminals a and b are open-circuited, the resulting return difference matrix is exactly the same as that found under normal operating conditions, and we have

$$\mathbf{F}(\text{input open-circuited}) = \mathbf{F}(\mathbf{X}) = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{16.49}$$

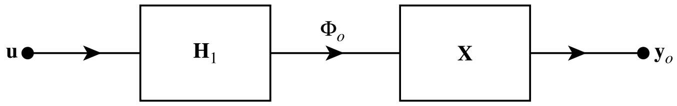

FIGURE 16.7 The block diagram of a multivariable open-loop control system.

Because

$$\mathbf{F}(\mathbf{X}) = \mathbf{1}_2 - \mathbf{AX} \tag{16.50}$$

the elements F_{11} and F_{21} are calculated with $y_{12} = 0$, whereas F_{12} and F_{22} are evaluated with $y_{21} = 0$. When the input terminals a and b are short circuited, the feedback loop is interrupted and only the second row and first column element of the matrix **A** is nonzero, and we obtain

$$\det \mathbf{F}(\text{input short-circuited}) = 1 \tag{16.51}$$

Because **X** is diagonal, the return difference function $F(y_{21})$ can be expressed in terms of det **F**(**X**) and the cofactor of the first row and first column element of **F**(**X**):

$$F(y_{21}) = \frac{\det \mathbf{F}(\mathbf{X})}{F_{22}} \tag{16.52}$$

Substituting these in (16.48) yields

$$F(y_{12})\big|_{y_{21}=0} F(y_{21}) = \frac{z_{aa,bb}(0,0)}{z_{aa,bb}(y_{12}, y_{21})} \tag{16.53}$$

where

$$F_{22} = 1 - a_{22} y_{12}\big|_{y_{21}=0} = F(y_{12})\big|_{y_{21}=0} \tag{16.54}$$

and a_{22} is the second row and second column element of **A**. Formula (16.53) was derived earlier in (15.7) using the network arrangements of Figures 15.7 and 15.8 to measure the elements $F(y_{12})|_{y_{21}=0}$ and $z_{aa,bb}(0,0)$, respectively.

16.5 The Sensitivity Matrix

We have studied the sensitivity of a transfer function with respect to the change of a particular element in the network. In a multiple-loop feedback network, we are usually interested in the sensitivity of a transfer function with respect to the variation of a set of elements in the network. This set may include either elements that are inherently sensitive to variation or elements where the effect on the overall amplifier performance is of paramount importance to the designers. For this, we introduce a sensitivity matrix and develop formulas for computing multiparameter sensitivity function for a multiple-loop feedback amplifier [7].

Figure 16.7 is the block diagram of a multivariable open-loop control system with n inputs and m outputs, whereas Figure 16.8 is the general feedback structure. If all feedback signals are obtainable from the output and if the controllers are linear, no loss of generality occurs by assuming the controller to be of the form given in Figure 16.9.

Denote the set of Laplace-transformed input signals by the n-vector **u**, the set of inputs to the network **X** in the open-loop configuration of Figure 16.7 by the p-vector $\mathbf{\Phi}_o$, and the set of outputs of the network

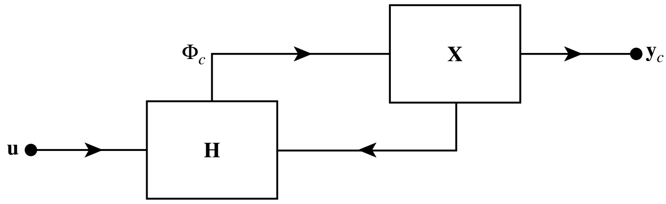

FIGURE 16.8 The general feedback structure.

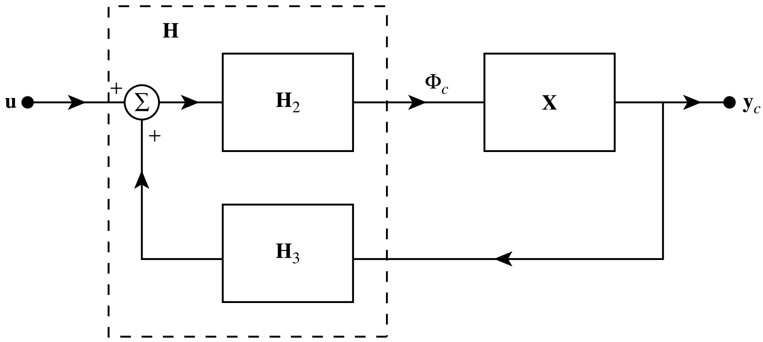

FIGURE 16.9 The general feedback configuration.

$\mathbf{X}$ of Figure 16.7 by the m-vector $\mathbf{y}_o$. Let the corresponding signals for the closed-loop configuration of Figure 16.9 be denoted by the n-vector $\mathbf{u}$, the p-vector $\mathbf{\Phi}_c$, and the m-vector $\mathbf{y}_c$, respectively. Then, from Figures 16.7 and 16.9, we obtain the following relations:

$$\mathbf{y}_o = \mathbf{X}\mathbf{\Phi}_o \tag{16.55a}$$

$$\mathbf{\Phi}_o = \mathbf{H}_1\mathbf{u} \tag{16.55b}$$

$$\mathbf{y}_c = \mathbf{X}\mathbf{\Phi}_c \tag{16.55c}$$

$$\mathbf{\Phi}_c = \mathbf{H}_2(\mathbf{u} + \mathbf{H}_3\mathbf{y}_c) \tag{16.55d}$$

where the transfer-function matrices $\mathbf{X}$, $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ are of order $m \times p$, $p \times n$, $p \times n$ and $n \times m$, respectively. Combining (16.55c) and (16.55d) yields

$$(\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3)\mathbf{y}_c = \mathbf{X}\mathbf{H}_2\mathbf{u} \tag{16.56}$$

or

$$\mathbf{y}_c = (\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3)^{-1}\,\mathbf{X}\mathbf{H}_2\mathbf{u} \tag{16.57}$$

The closed-loop transfer function matrix $\mathbf{W}(\mathbf{X})$ that relates the input vector $\mathbf{u}$ to the output vector $\mathbf{y}_c$ is defined by the equation

$$\mathbf{y}_c = \mathbf{W}(\mathbf{X})\mathbf{u} \tag{16.58}$$

identifying from (16.57) the $m \times n$ matrix

$$\mathbf{W}(\mathbf{X}) = \left(\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}\mathbf{H}_2 \tag{16.59}$$

Now, suppose that **X** is perturbed from **X** to **X** + **δX**. The outputs of the open-loop and closed-loop systems of Figure 16.7 and 16.9 will no longer be the same as before. Distinguishing the new from the old variables by the superscript +, we have

$$\mathbf{y}_o^+ = \mathbf{X}^+\mathbf{\Phi}_o \tag{16.60a}$$

$$\mathbf{y}_c^+ = \mathbf{X}^+\mathbf{\Phi}_c^+ \tag{16.60b}$$

$$\mathbf{\Phi}_c^+ = \mathbf{H}_2\left(\mathbf{u} + \mathbf{H}_3\mathbf{y}_c^+\right) \tag{16.60c}$$

where $\mathbf{\Phi}_o$ remains the same.

We next proceed to compare the relative effects of the variations of **X** on the performance of the open-loop and the closed-loop systems. For a meaningful comparison, we assume that $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ are such that when there is no variation of **X**, $\mathbf{y}_o = \mathbf{y}_c$. Define the error vectors resulting from perturbation of **X** as

$$\mathbf{E}_o = \mathbf{y}_o - \mathbf{y}_o^+ \tag{16.61a}$$

$$\mathbf{E}_c = \mathbf{y}_c - \mathbf{y}_c^+ \tag{16.61b}$$

A square matrix relating $\mathbf{E}_o$ to $\mathbf{E}_c$ is called the **sensitivity matrix** $\mathcal{S}(\mathbf{X})$ for the transfer function matrix $\mathbf{W}(\mathbf{X})$ with respect to the variations of **X**:

$$\mathbf{E}_c = \mathcal{S}(\mathbf{X})\mathbf{E}_o \tag{16.62}$$

In the following, we express the **sensitivity matrix** $\mathcal{S}(\mathbf{X})$ in terms of the system matrices **X**, $\mathbf{H}_2$, and $\mathbf{H}_3$.

The input and output relation similar to that given in (16.57) for the perturbed system can be written as

$$\mathbf{y}_c^+ = \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}^+\mathbf{H}_2\mathbf{u} \tag{16.63}$$

Substituting (16.57) and (16.63) in (16.61b) gives

$$\begin{aligned}
\mathbf{E}_c = \mathbf{y}_c - \mathbf{y}_c^+ &= \left[\left(\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}\mathbf{H}_2 - \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}^+\mathbf{H}_2\right]\mathbf{u} \\
&= \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\left\{\left[\mathbf{1}_m - (\mathbf{X} + \delta\mathbf{X})\mathbf{H}_2\mathbf{H}_3\right]\left(\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}\mathbf{H}_2 - (\mathbf{X} + \delta\mathbf{X})\mathbf{H}_2\right\}\mathbf{u} \\
&= \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\left[\mathbf{X}\mathbf{H}_2 - \delta\mathbf{X}\mathbf{H}_2\mathbf{H}_3\left(\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{X}\mathbf{H}_2 - \mathbf{X}\mathbf{H}_2 - \delta\mathbf{X}\mathbf{H}_2\right]\mathbf{u} \\
&= -\left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\delta\mathbf{X}\mathbf{H}_2\left[\mathbf{1}_n + \mathbf{H}_3\mathbf{W}(\mathbf{X})\right]\mathbf{u}
\end{aligned} \tag{16.64}$$

From (16.55d) and (16.58), we obtain

$$\mathbf{\Phi}_c = \mathbf{H}_2\left[\mathbf{1}_n + \mathbf{H}_3\mathbf{W}(\mathbf{X})\right]\mathbf{u} \tag{16.65}$$

Because by assuming that $\mathbf{y}_o = \mathbf{y}_c$, we have

$$\mathbf{\Phi}_o = \mathbf{\Phi}_c = \mathbf{H}_2\left[\mathbf{1}_n + \mathbf{H}_3\mathbf{W}(\mathbf{X})\right]\mathbf{u} \tag{16.66}$$

yielding

$$\mathbf{E}_o = \mathbf{y}_o - \mathbf{y}_o^+ = \left(\mathbf{X} - \mathbf{X}^+\right)\mathbf{\Phi}_o = -\boldsymbol{\delta}\mathbf{X}\mathbf{H}_2\left[\mathbf{1}_n + \mathbf{H}_3\mathbf{W}(\mathbf{X})\right]\mathbf{u} \tag{16.67}$$

Combining (16.64) and (16.67) yields an expression relating the error vectors $\mathbf{E}_c$ and $\mathbf{E}_o$ of the closed-loop and open-loop systems by

$$\mathbf{E}_c = \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1}\mathbf{E}_o \tag{16.68}$$

obtaining the sensitivity matrix as

$$\mathcal{S}(\mathbf{X}) = \left(\mathbf{1}_m - \mathbf{X}^+\mathbf{H}_2\mathbf{H}_3\right)^{-1} \tag{16.69}$$

For small variations of $\mathbf{X}$, $\mathbf{X}^+$ is approximately equal to $\mathbf{X}$. Thus, in Figure 16.9, if the matrix triple product $\mathbf{X}\mathbf{H}_2\mathbf{H}_3$ is regarded as the *loop-transmission matrix* and $-\mathbf{X}\mathbf{H}_2\mathbf{H}_3$ as the *return ratio matrix,* then the difference between the unit matrix and the loop-transmission matrix,

$$\mathbf{1}_m - \mathbf{X}\mathbf{H}_2\mathbf{H}_3 \tag{16.70}$$

can be defined as the *return difference matrix.* Therefore, (16.69) is a direct extension of the sensitivity function defined for a single-input, single-output system and for a single parameter. Recall that in (14.33) we demonstrated that, using the ideal feedback model, the sensitivity function of the closed-loop transfer function with respect to the forward amplifier gain is equal to the reciprocal of its return difference with respect to the same parameter.

In particular, when $\mathbf{W}(\mathbf{X})$, $\boldsymbol{\delta}\mathbf{X}$, and $\mathbf{X}$ are square and nonsingular, from (16.55a), (16.55b), and (16.58), (16.61) can be rewritten as

$$\mathbf{E}_c = \mathbf{y}_c - \mathbf{y}_c^+ = \left[\mathbf{W}(\mathbf{X}) - \mathbf{W}^+(\mathbf{X})\right]\mathbf{u} = -\boldsymbol{\delta}\mathbf{W}(\mathbf{X})\mathbf{u} \tag{16.71a}$$

$$\mathbf{E}_o = \mathbf{y}_o - \mathbf{y}_o^+ = \left[\mathbf{X}\mathbf{H}_1 - \mathbf{X}^+\mathbf{H}_1\right]\mathbf{u} = -\boldsymbol{\delta}\mathbf{X}\mathbf{H}_1\mathbf{u} \tag{16.71b}$$

If $\mathbf{H}_1$ is nonsingular, $\mathbf{u}$ in (16.71b) can be solved for and substituted in (16.71a) to give

$$\mathbf{E}_c = \boldsymbol{\delta}\mathbf{W}(\mathbf{X})\mathbf{H}_1^{-1}(\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{E}_o \tag{16.72}$$

As before, for meaningful comparison, we require that $\mathbf{y}_o = \mathbf{y}_c$ or

$$\mathbf{X}\mathbf{H}_1 = \mathbf{W}(\mathbf{X}) \tag{16.73}$$

From (16.72), we obtain

$$\mathbf{E}_c = \boldsymbol{\delta}\mathbf{W}(\mathbf{X})\mathbf{W}^{-1}(\mathbf{X})\mathbf{X}(\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{E}_o \tag{16.74}$$

identifying that

$$\mathcal{S}(\mathbf{X}) = \boldsymbol{\delta}\mathbf{W}(\mathbf{X})\mathbf{W}^{-1}(\mathbf{X})\mathbf{X}(\boldsymbol{\delta}\mathbf{X})^{-1} \tag{16.75}$$

This result is to be compared with the scalar sensitivity function defined in (14.26), which can be put in the form

$$\mathcal{S}(x) = (\delta w)w^{-1}x(\delta x)^{-1} \tag{16.76}$$

16.6 Multiparameter Sensitivity

In this section, we derive formulas for the effect of change of **X** on a scalar transfer function $w(\mathbf{X})$.

Let x_k, $k = 1, 2, \ldots, pq$, be the elements of **X**. The multivariable Taylor series expansion of $w(\mathbf{X})$ with respect to x_k is given by

$$\delta w = \sum_{k=1}^{pq} \frac{\partial w}{\partial x_k} \delta x_k + \sum_{j=1}^{pq} \sum_{k=1}^{pq} \frac{\partial^2 w}{\partial x_j \partial x_k} \frac{\delta x_j \delta x_k}{2!} + \cdots \tag{16.77}$$

The first-order perturbation can then be written as

$$\delta w \approx \sum_{k=1}^{pq} \frac{\partial w}{\partial x_k} \delta x_k \tag{16.78}$$

Using (14.26), we obtain

$$\frac{\delta w}{w} \approx \sum_{k=1}^{pq} \mathcal{S}(x_k) \frac{\delta x_k}{x_k} \tag{16.79}$$

This expression gives the fractional change of the transfer function w in terms of the scalar sensitivity functions $\mathcal{S}(x_k)$.

Refer to the fundamental matrix feedback-flow graph of Figure 16.3. If the amplifier has a single input and a single output from (16.35), the overall transfer function $w(\mathbf{X})$ of the multiple-loop feedback amplifier becomes

$$w(\mathbf{X}) = D + \mathbf{C}\mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\mathbf{B} \tag{16.80}$$

When **X** is perturbed to $\mathbf{X}^+ = \mathbf{X} + \boldsymbol{\delta}\mathbf{X}$, the corresponding expression of (16.80) is given by

$$w(\mathbf{X}) + \delta w(\mathbf{X}) = D + \mathbf{C}(\mathbf{X} + \boldsymbol{\delta}\mathbf{X})(\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{B} \tag{16.81}$$

or

$$\delta w(\mathbf{X}) = \mathbf{C}\left[(\mathbf{X} + \boldsymbol{\delta}\mathbf{X})(\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})^{-1} - \mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\right]\mathbf{B} \tag{16.82}$$

As $\boldsymbol{\delta}\mathbf{X}$ approaches zero, we obtain

$$\begin{aligned}
\delta w(\mathbf{X}) &= \mathbf{C}\left[(\mathbf{X} + \boldsymbol{\delta}\mathbf{X}) - \mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}(\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})\right](\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{B} \\
&= \mathbf{C}\left[\boldsymbol{\delta}\mathbf{X} + \mathbf{X}(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\mathbf{A}\boldsymbol{\delta}\mathbf{X}\right](\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{B} \\
&= \mathbf{C}(\mathbf{1}_q - \mathbf{X}\mathbf{A})^{-1}(\boldsymbol{\delta}\mathbf{X})(\mathbf{1}_p - \mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\delta}\mathbf{X})^{-1}\mathbf{B} \\
&\approx \mathbf{C}(\mathbf{1}_q - \mathbf{X}\mathbf{A})^{-1}(\boldsymbol{\delta}\mathbf{X})(\mathbf{1}_p - \mathbf{A}\mathbf{X})^{-1}\mathbf{B}
\end{aligned} \tag{16.83}$$

where **C** is a row q vector and **B** is a column p vector. Write

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_q \end{bmatrix} \tag{16.84a}$$

$$\mathbf{B}' = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} \tag{16.84b}$$

$$\tilde{\mathbf{W}} = \mathbf{X}(\mathbf{1}_p - \mathbf{AX})^{-1} = (\mathbf{1}_q - \mathbf{XA})^{-1}\mathbf{X} = [\tilde{w}_{ij}] \tag{16.84c}$$

The increment $\delta w(\mathbf{X})$ can be expressed in terms of the elements of (16.84) and those of $\mathbf{X}$. In the case where $\mathbf{X}$ is diagonal with

$$\mathbf{X} = \text{diag}\begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} \tag{16.85}$$

where $p = q$, the expression for $\delta w(\mathbf{X})$ can be succinctly written as

$$\begin{aligned} \delta w(\mathbf{X}) &= \sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{j=1}^{p} c_i\left(\frac{\tilde{w}_{ik}}{x_k}\right)(\delta x_k)\left(\frac{\tilde{w}_{kj}}{x_k}\right)b_j \\ &= \sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{j=1}^{p} \frac{c_i\tilde{w}_{ik}\tilde{w}_{kj}b_j}{x_k}\,\frac{\delta x_k}{x_k} \end{aligned} \tag{16.86}$$

Comparing this with (16.79), we obtain an explicit form for the single-parameter sensitivity function as

$$\mathcal{S}(x_k) = \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{c_i\tilde{w}_{ik}\tilde{w}_{kj}b_j}{x_k w(\mathbf{X})} \tag{16.87}$$

Thus, knowing (16.84) and (16.85), we can calculate the multiparameter sensitivity function for the scalar transfer function $w(\mathbf{X})$ immediately.

Example 6. Consider again the voltage-series feedback amplifier of Figure 13.9, an equivalent network of which is shown in Figure 16.4. Assume that V_s is the input and V_{25} the output. The transfer function of interest is the amplifier voltage gain V_{25}/V_s. The elements of main concern are the two controlling parameters of the controlled sources. Thus, we let

$$\mathbf{X} = \begin{bmatrix} \tilde{\alpha}_1 & 0 \\ 0 & \tilde{\alpha}_2 \end{bmatrix} = \begin{bmatrix} 0.0455 & 0 \\ 0 & 0.0455 \end{bmatrix} \tag{16.88}$$

From (16.27) we have

$$\mathbf{A} = \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix} \tag{16.89a}$$

$$\mathbf{B}' = \begin{bmatrix} 0.91748 & 0 \end{bmatrix} \tag{16.89b}$$

$$\mathbf{C} = \begin{bmatrix} 45.391 & -2372.32 \end{bmatrix} \tag{16.89c}$$

yielding

$$\tilde{\mathbf{W}} = \mathbf{X}(\mathbf{1}_2 - \mathbf{AX})^{-1} = 10^{-4}\begin{bmatrix} 4.85600 & 10.02904 \\ -208.245 & 24.91407 \end{bmatrix} \tag{16.90}$$

Also, from (16.13) we have

$$w(\mathbf{X}) = \frac{V_{25}}{V_s} = 45.387 \tag{16.91}$$

To compute the sensitivity functions with respect to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, we apply (16.87) and obtain

$$\mathcal{S}(\tilde{\alpha}_1) = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{c_i \tilde{w}_{i1} \tilde{w}_{1j} b_j}{\tilde{\alpha}_1 w(\mathbf{X})} = \frac{c_1 \tilde{w}_{11} \tilde{w}_{11} b_1 + c_1 \tilde{w}_{11} \tilde{w}_{12} b_2 + c_2 \tilde{w}_{21} \tilde{w}_{11} b_1 + c_2 \tilde{w}_{21} \tilde{w}_{12} b_2}{\tilde{\alpha}_1 w} = 0.01066 \tag{16.92a}$$

$$\mathcal{S}(\tilde{\alpha}_2) = \frac{c_1 \tilde{w}_{12} \tilde{w}_{21} b_1 + c_1 \tilde{w}_{12} \tilde{w}_{22} b_2 + c_2 \tilde{w}_{22} \tilde{w}_{21} b_1 + c_2 \tilde{w}_{22} \tilde{w}_{22} b_2}{\tilde{\alpha}_2 w} = 0.05426 \tag{16.92b}$$

As a check, we use (14.30) to compute these sensitivities. From (13.45) and (13.52), we have

$$F(\tilde{\alpha}_1) = 93.70 \tag{16.93a}$$

$$F(\tilde{\alpha}_2) = 18.26 \tag{16.93b}$$

$$\hat{F}(\tilde{\alpha}_1) = 103.07 \times 10^3 \tag{16.93c}$$

$$\hat{F}(\tilde{\alpha}_2) = 2018.70 \tag{16.93d}$$

Substituting these in (14.30) the sensitivity functions are:

$$\mathcal{S}(\tilde{\alpha}_1) = \frac{1}{F(\tilde{\alpha}_1)} - \frac{1}{\hat{F}(\tilde{\alpha}_1)} = 0.01066 \tag{16.94a}$$

$$\mathcal{S}(\tilde{\alpha}_2) = \frac{1}{F(\tilde{\alpha}_2)} - \frac{1}{\hat{F}(\tilde{\alpha}_2)} = 0.05427 \tag{16.94b}$$

confirming (16.92).

Suppose that $\tilde{\alpha}_1$ is changed by 4% and $\tilde{\alpha}_2$ by 6%. The fractional change of the voltage gain $w(\mathbf{X})$ is found from (16.79) as

$$\frac{\delta w}{w} \approx \mathcal{S}(\tilde{\alpha}_1)\frac{\delta\tilde{\alpha}_1}{\tilde{\alpha}_1} + \mathcal{S}(\tilde{\alpha}_2)\frac{\delta\tilde{\alpha}_2}{\tilde{\alpha}_2} = 0.003683 \tag{16.95}$$

or 0.37%.

References

[1] F. H. Blecher, "Design principles for single loop transistor feedback amplifiers," *IRE Trans. Circuit Theory,* vol. CT-4, pp. 145–156, 1957.

[2] H. W. Bode, *Network Analysis and Feedback Amplifier Design,* Princeton, NJ: Van Nostrand, 1945.

[3] W.-K. Chen, "Indefinite-admittance matrix formulation of feedback amplifier theory," *IEEE Trans. Circuits Syst.,* vol. CAS-23, pp. 498–505, 1976.

[4] W.-K. Chen, "On second-order cofactors and null return difference in feedback amplifier theory," *Int. J. Circuit Theory Appl.,* vol. 6, pp. 305–312, 1978.

[5] W.-K. Chen, *Active Network and Feedback Amplifier Theory,* New York: McGraw-Hill, 1980, chaps. 2, 4, 5, 7.
[6] W.-K. Chen, *Active Network and Analysis,* Singapore: World Scientific, 1991, chaps. 2, 4, 5, 7.
[7] J. B. Cruz, Jr. and W. R. Perkins, "A new approach to the sensitivity problem in multivariable feedback system design," *IEEE Trans. Autom. Control,* vol. AC-9, pp. 216–223, 1964.
[8] S. S. Haykin, *Active Network Theory*, Reading, MA: Addison-Wesley, 1970.
[9] E. S. Kuh and R. A. Rohrer, *Theory of Linear Active Networks,* San Francisco: Holden-Day, 1967.
[10] I. W. Sandberg, "On the theory of linear multi-loop feedback systems," *Bell Syst. Tech. J.,* vol. 42, pp. 355–382, 1963.