# 3

# Terminal and Port Representations

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## 3.1 Introduction

Frequently, it is useful to decompose a large circuit into subcircuits and to consider the subcircuits separately. Subcircuits are connected to other subcircuits using terminals or ports. Terminal and port representations of a subcircuit describe how that subcircuit will act when connected to other subcircuits. Terminal and port representations do not provide the details of mesh or node equations because these details are not required to describe how a subcircuit interacts with other subcircuits.

In this chapter, terminal and port representations of circuits are described. Particular attention is given to the distinction between terminals and ports. Applications show the usefulness of terminal and port representations.

## 3.2 Terminal Representations

Figure 3.1 illustrates a subcircuit that can be connected to other subcircuits using terminals. The subcircuit is shown symbolically on the left and an example is shown on the right. Nodes *a*, *b*, *c*, and *d* are terminals and may be used to connect this circuit to other circuits. Node *e* is internal to the circuit and is not a terminal. The terminal voltages $V_a$, $V_b$, $V_c$, and $V_d$ are node voltages with respect to an arbitrary reference node. The terminal currents $I_a$, $I_b$, $I_c$, and $I_d$ describe currents that will exist when this subcircuit is connected to other subcircuits. Terminal representations show how the terminal voltages and currents are related. Several equivalent representations are possible, depending on which of the terminal currents and voltages are selected as independent variables.

A terminal representation of the example network in Figure 3.1 can be obtained by writing a node equation for the network. A simpler procedure is available for passive networks, i.e., networks consisting entirely of resistors, capacitors, and inductors. Because the circuit in Figure 3.1 is a passive circuit, the simpler procedure will be used to write terminal equations to represent this circuit.

The nodal admittance matrix of the example circuit will have five rows and five columns. The rows, and also the columns, of this matrix will correspond to the five nodes of the circuit in the order *a*, *b*, *c*, *d* and *e*. For example, the fourth row of the nodal admittance matrix corresponds to node *d* and the third column corresponds to node *c*. Let $y_{ij}$ denote the admittance of a branch of the network which is incident with nodes *i* and *j* and let $y_{ij}$ denote the element of the nodal admittance matrix that is in row *i* and column *j*. Then,

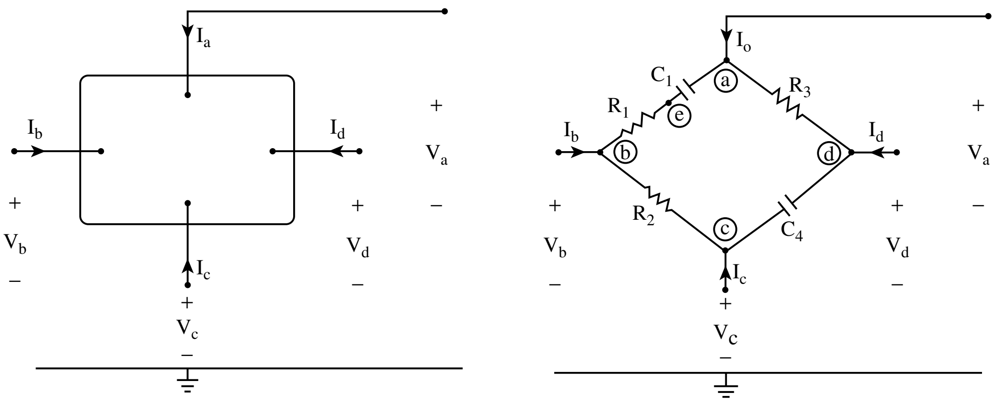



**FIGURE 3.1** A four-terminal network.

$$Y_{ii} = \sum_{\substack{\text{all branches} \\ \text{incident to node } i}} y_{ik} \tag{3.1}$$

i.e., the diagonal element of the nodal admittance matrix in row $i$ and column $i$ is equal to the sum of the admittances of all branches in the circuit that are incident with node $i$. The off-diagonal elements of the nodal admittance matrix are given by

$$Y_{ij} = -\sum_{\substack{\text{all branches incident} \\ \text{to both nodes } i \text{ and } j}} y_{ij} \tag{3.2}$$

The example circuit is represented by the node equation

$$\begin{pmatrix} I_a \\ I_b \\ I_c \\ I_d \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 s + \frac{1}{R_3} & 0 & 0 & -\frac{1}{R_3} & -C_1 s \\ 0 & \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} & 0 & -\frac{1}{R_1} \\ 0 & -\frac{1}{R_2} & C_4 s + \frac{1}{R_2} & -C_4 s & 0 \\ -\frac{1}{R_3} & 0 & -C_4 s & C_4 s + \frac{1}{R_3} & 0 \\ -C_1 s & -\frac{1}{R_1} & 0 & 0 & C_1 s + \frac{1}{R_1} \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{pmatrix} \tag{3.3}$$

Suppose that $C_1 = 1$ F, $C_4 = 2$ F, $R_1 = 1/2\ \Omega$, $R_2 = 1/4\ \Omega$, and $R_3 = 1\ \Omega$. Then,

$$\begin{pmatrix} I_a \\ I_b \\ I_c \\ I_d \\ 0 \end{pmatrix} = \begin{pmatrix} s+1 & 0 & 0 & -1 & -1s \\ 0 & 6 & -4 & 0 & -2 \\ 0 & -4 & 2s+4 & -2s & 0 \\ -1 & 0 & -2s & 2s+1 & 0 \\ -s & -2 & 0 & 0 & s+2 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{pmatrix}$$

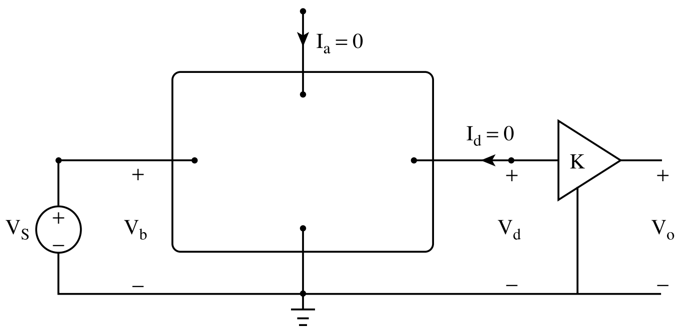



**FIGURE 3.2** First application of the four-terminal network.

No external current $I_e$ exists because node $e$ is not a terminal. Row 5 of this equation, corresponding to node $e$, can be solved for $V_e$ and then $V_e$ can be eliminated. Doing so results in

$$\begin{pmatrix} I_a \\ I_b \\ I_c \\ I_d \end{pmatrix} = \begin{pmatrix} \frac{3s+2}{s+2} & -\frac{2s}{s+2} & 0 & -1 \\ -\frac{2s}{s+2} & \frac{6s+8}{s+2} & -4 & 0 \\ 0 & -4 & 2s+4 & -2s \\ -1 & 0 & -2s & 2s+1 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \end{pmatrix} \tag{3.4}$$

The terminal voltages were selected to be the independent variables and the entries in the matrix are admittances. Because none of the nodes of the subcircuit was chosen to be the reference node, the rows and columns of the matrix both sum to zero. This matrix is called an indefinite admittance matrix. Because it is singular, Eq. (3.4) cannot be solved directly. To see the utility of Eq. (3.4), consider Figure 3.2. Here, the four-terminal network has been connected to a voltage source and an amplifier and node $c$ has been grounded. This external circuitry restricts the terminal currents and voltages of the four terminal network. In particular,

$$V_b = V_s, \quad I_a = 0, \quad V_c = 0, \quad I_d = 0 \quad \text{and} \quad V_o = KV_d \tag{3.5}$$

Under these conditions, Eq. (3.4) becomes

$$\begin{pmatrix} 0 \\ I_b \\ I_c \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3s+2}{s+2} & -\frac{2s}{s+2} & 0 & -1 \\ -\frac{2s}{s+2} & \frac{6s+8}{s+2} & -4 & 0 \\ 0 & -4 & 2s+4 & -2s \\ -1 & 0 & -2s & 2s+1 \end{pmatrix} \begin{pmatrix} V_a \\ V_s \\ 0 \\ \frac{V_o}{K} \end{pmatrix} \tag{3.6}$$

Because $I_b$ and $I_c$ are not of interest, the second and third rows of this equation can be ignored. The first and fourth rows can be solved to obtain the transfer function of this circuit

$$\frac{V_o(s)}{V_s(s)} = \frac{K}{3(s+1)} \tag{3.7}$$

Next, consider Figure 3.3. The four terminal network is used in a second, different application. In this case,

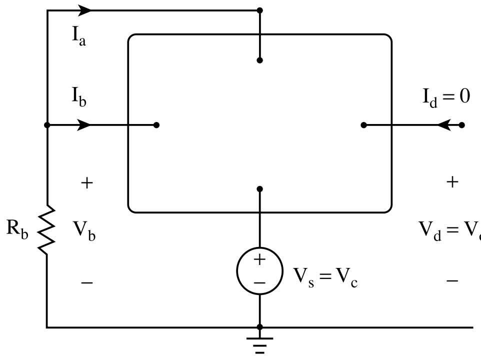



**FIGURE 3.3** Second application of the four-terminal network.

$$I_a + I_b = -\frac{V_b}{R_b}, \quad V_c = V_s, \quad V_a = V_b \quad I_d = 0 \quad \text{and} \quad V_d = V_o \tag{3.8}$$

Because $V_a = V_b$, the first two columns of Eq. (3.4) can be added together, replacing $V_a$ by $V_b$. Also, it is convenient to add the first two rows together to obtain $I_a + I_b$. Then, Eq. (3.4) reduces to

$$\begin{pmatrix} -\dfrac{V_b}{R_b} \\ I_c \\ 0 \end{pmatrix} = \begin{pmatrix} 5 & -4 & -1 \\ -4 & 2s+4 & -2s \\ -1 & -2s & 2s+1 \end{pmatrix} \begin{pmatrix} V_b \\ V_s \\ V_o \end{pmatrix} \tag{3.9}$$

Because $I_c$ is not of interest, the second row of this equation can be ignored. The first and third rows of this equation can be solved to obtain the transfer function

$$\frac{V_o(s)}{V_s(s)} = \frac{(10R_b + 2)s + 4R_b}{(10R_b + 2)s + 4R_b + 1} \tag{3.10}$$

These examples illustrate the utility of the terminal equations. In these examples, the problem of analyzing a network was divided into two parts. First, the terminal equation was obtained from the node equation by eliminating the rows and columns corresponding to nodes that are not terminals. Second, the terminal equation is combined with equations describing the external circuitry connected to the four terminal network. When the external circuit is changed, only the second step must be redone. The advantage of representing a subnetwork by terminal equations is greater when

1. The subnetwork has many nodes that are not terminals.
2. The subnetwork is expected to be a component of many different networks.

## 3.3 Port Representations

A port consists of two terminals with the restriction that the terminal currents have the same magnitude but opposite sign. Figure 3.4 shows a two-port network. In this case, the two-port network was constructed from the four-terminal network by pairing terminals $a$ and $b$ form port 1 and pairing terminals $d$ and $c$ to form port 2. The restrictions

$$I_1 = I_a = -I_b \quad \text{and} \quad I_2 = I_d = -I_c \tag{3.11}$$

must be satisfied in order for these two pairs of terminals to be ports.

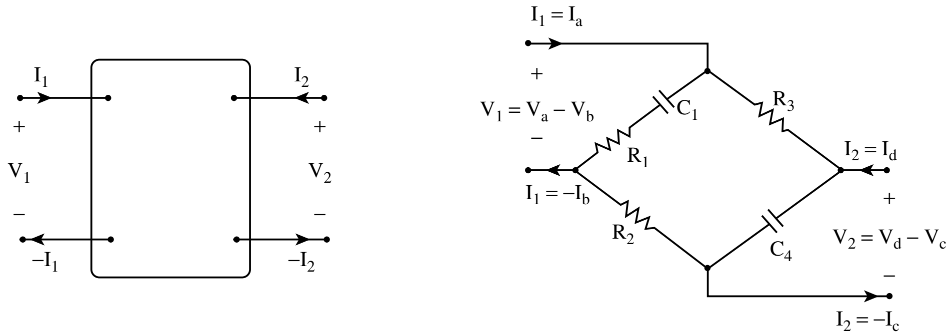



**FIGURE 3.4** A two-port network.

The behavior of the two-port network is described using four variables. These variables are the port voltages, $V_1$ and $V_2$ and the port currents $I_1$, and $I_2$. Several equivalent representations are possible, depending on which two of the port currents and voltages are selected as independent variables. The left column of Table 3.1 shows the two-port representations corresponding to four of the possible choices of independent variables. These representations are equivalent, and the right column of Table 3.1 shows how one representation can be obtained from another.

In row 1 of Table 3.1, the port voltages are selected to be the independent variables. In this case, the two-port circuit is represented by an equation of the form

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{3.12}$$

The elements of the matrix in this equation have units of admittance. They are denoted using the letter $y$ to suggest admittance and are called the "$y$ parameters" or the "admittance parameters" of the two-port network.

In row 2 of Table 3.1, the port currents are selected to be the independent variables. Now the elements of the matrix have units of impedance. They are denoted using the letter $z$ to suggest impedance and are called the "$z$ parameters" or the "impedance parameters" of the two-port network.

In row 3 of Table 3.1, $I_1$ and $V_2$ are the independent variables. The elements of the matrix do not have the same units. In this sense, this is a hybrid matrix. They are denoted using the letter $h$ to suggest hybrid and are called the "$h$ parameters" or the "hybrid parameters" of the two-port network. Hybrid parameters are frequently used to represent bipolar transistors.

In the last row of Table 3.1, the independent signals are $V_2$ and $-I_2$. In this case, the two-port parameters are called "transmission parameters" or "*ABCD* parameters". They are convenient when two-port networks are connected in cascade.

Next, consider the problem of calculating or measuring the parameters of a two-port circuit. Equation (3.12) suggests a procedure for calculating the $y$ parameters of a two-port circuit. The first row of Eq. (3.12) is

$$I_1 = y_{11}V_1 + y_{12}V_2 \tag{3.13}$$

Setting $V_1 = 0$ leads to

$$y_{12} = \frac{I_1}{V_2} \quad \text{when} \quad V_1 = 0 \tag{3.14}$$

**TABLE 3.1** Relationships between Two-Port Representations

| | |
|---|---|
| $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ <br> $\lvert Y\rvert = y_{11}y_{22} - y_{12}y_{21}$ | $\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} \frac{z_{22}}{\lvert Z\rvert} & -\frac{z_{12}}{\lvert Z\rvert} \\ -\frac{z_{21}}{\lvert Z\rvert} & \frac{z_{11}}{\lvert Z\rvert} \end{pmatrix} = \begin{pmatrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\lvert H\rvert}{h_{11}} \end{pmatrix} = \begin{pmatrix} \frac{D}{B} & -\frac{\lvert T\rvert}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{pmatrix}$ |
| $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ <br> $\lvert Z\rvert = z_{11}z_{22} - z_{12}z_{21}$ | $\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} \frac{y_{22}}{\lvert Y\rvert} & -\frac{y_{12}}{\lvert Y\rvert} \\ -\frac{y_{21}}{\lvert Y\rvert} & \frac{y_{11}}{\lvert Y\rvert} \end{pmatrix} = \begin{pmatrix} \frac{\lvert H\rvert}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{pmatrix} = \begin{pmatrix} \frac{A}{C} & -\frac{\lvert T\rvert}{C} \\ \frac{1}{C} & \frac{D}{C} \end{pmatrix}$ |
| $\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$ <br> $\lvert H\rvert = h_{11}h_{22} - h_{12}h_{21}$ | $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{y_{11}} & -\frac{y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\lvert Y\rvert}{y_{11}} \end{pmatrix} = \begin{pmatrix} \frac{\lvert Z\rvert}{z_{22}} & \frac{z_{12}}{z_{22}} \\ -\frac{z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{pmatrix} = \begin{pmatrix} \frac{B}{D} & \frac{\lvert T\rvert}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{pmatrix}$ |
| $\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}$ <br> $\lvert T\rvert = AD - BC$ | $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{y_{22}}{y_{21}} & -\frac{1}{y_{21}} \\ -\frac{\lvert Y\rvert}{y_{21}} & -\frac{y_{11}}{y_{21}} \end{pmatrix} = \begin{pmatrix} \frac{z_{11}}{z_{21}} & \frac{\lvert Z\rvert}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{pmatrix} = \begin{pmatrix} -\frac{\lvert H\rvert}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{pmatrix}$ |

This equation describes a procedure that can be used to measure or calculate $y_{12}$. A short circuit is connected to port 1 to set $V_1 = 0$. A voltage source having voltage $V_2$ is connected across port 2 and the current, $I_1$, in the short circuit is calculated. Finally, $y_{12}$ is calculated as the ratio of $I_1$ to $V_2$.

Similar procedures can be used to calculate any of the $y$, $z$, $h$, or transmission parameters. Table 3.2 tabulates these procedures.

As an example of how Table 3.2 can be used, consider calculating the $y$ parameters of the two port circuit shown in Figure 3.3. Recall that $C_1 = 1$ F, $C_4 = 2$ F, $R_1 = 1/2\ \Omega$, $R_2 = 1/4\ \Omega$, and $R_3 = 1\ \Omega$.) According to Table 3.2, two cases will have to be considered. In the first, a voltage source is connected to port 1 and a short circuit is connected to port 2. In the second, a short circuit is connected across port 1 and a voltage source is connected to port 2. The resulting circuits are shown in Figure 3.5. In Figure 3.5(a)

$$I_1 = \frac{V_1}{R_1 + \frac{1}{C_1 s}} + \frac{V_1}{R_2 + R_3} = \frac{14s + 8}{5s + 10} V_1 = y_{11} V_1 \tag{3.15}$$

and

$$I_2 = -\frac{V_1}{R_2 + R_3} = -\frac{4}{5} V_1 = y_{21} V_1 \tag{3.16}$$

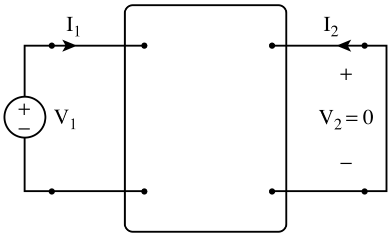
In Figure 3.5(b)

$$I_1 = -\frac{V_2}{R_2 + R_3} = -\frac{4}{5} V_2 = y_{12} V_2 \tag{3.17}$$

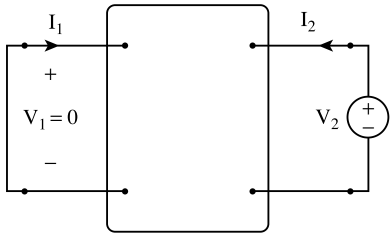
and

$$I_2 = \frac{V_2}{\frac{1}{C_4 s}} + \frac{V_2}{R_2 + R_3} = \frac{10s + 4}{5} V_2 = y_{22} V_2 \tag{3.18}$$

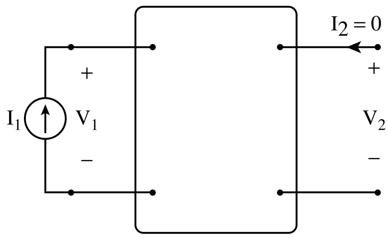
**TABLE 3.2** Calculation of Two-Port Parameters



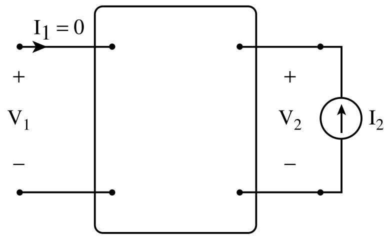

$y_{11} = \frac{I_1}{V_1}$ and $y_{21} = \frac{I_2}{V_1}$ when $V_2 = 0$



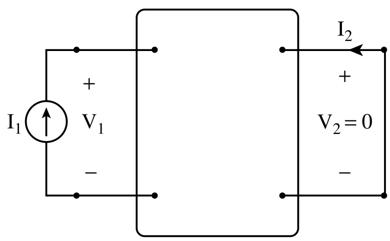

$y_{12} = \frac{I_1}{V_2}$ and $y_{22} = \frac{I_2}{V_2}$ when $V_1 = 0$



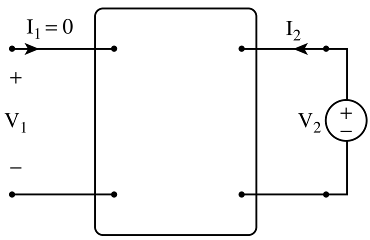

$z_{11} = \frac{V_1}{I_1}$ and $z_{21} = \frac{V_2}{I_1}$ when $I_2 = 0$



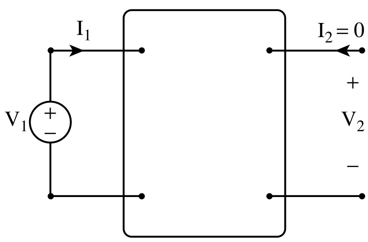

$z_{12} = \frac{V_1}{I_2}$ and $y_{22} = \frac{V_2}{I_1}$ when $I_1 = 0$




$h_{11} = \frac{V_1}{I_1}$ and $h_{21} = \frac{I_2}{I_1}$ when $V_2 = 0$

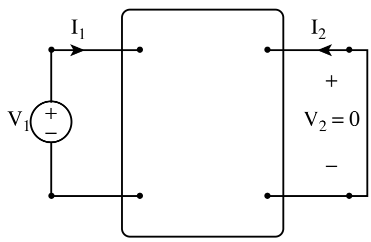



$h_{12} = \frac{V_1}{V_2}$ and $h_{22} = \frac{I_2}{V_2}$ when $I_1 = 0$




$A = \frac{V_1}{V_2}$ when $I_2 = 0$

$C = \frac{I_1}{V_2}$ when $I_2 = 0$




$B = \frac{V_1}{-I_2}$ when $V_2 = 0$

$D = \frac{I_1}{-I_2}$ when $V_2 = 0$

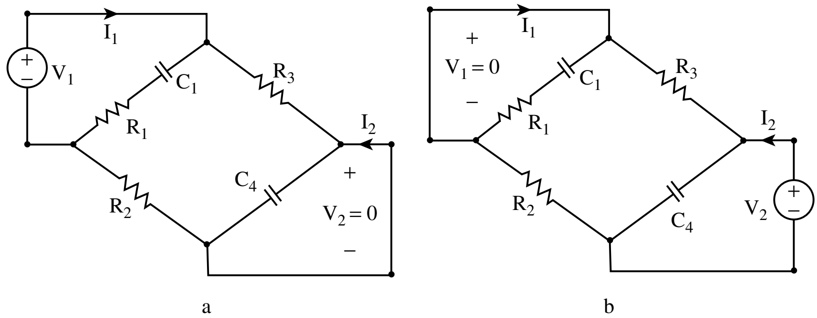



**FIGURE 3.5** The test circuits used to calculate the $y$ parameters of the example circuit.

Finally, the two-port network is represented by

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} \dfrac{14s+8}{5s+10} & -\dfrac{4}{5} \\ -\dfrac{4}{5} & \dfrac{10s+4}{5} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{3.19}$$

To continue the example, row 3 Table 3.1 illustrates how to convert these admittance parameters to hybrid parameters. From Table 3.1,

$$|Y| = \frac{14s+8}{5s+10}\left[\frac{10s+4}{5}\right] - \left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right) = \frac{28s^2+24s}{5(s+2)} \tag{3.20}$$

$$h_{11} = \frac{1}{y_{11}} = \frac{5s+10}{14s+8} \tag{3.21}$$

$$h_{12} = -\frac{y_{12}}{y_{11}} = \frac{2s+4}{7s+4} \tag{3.22}$$

$$h_{21} = \frac{y_{21}}{y_{11}} = -\frac{2s+4}{7s+4} \tag{3.23}$$

$$h_{22} = \frac{|Y|}{y_{11}} = \frac{(14s+12)s}{7s+4} \tag{3.24}$$

To complete the example, notice that Table 3.2 shows how to calculate the hybrid parameters directly from the circuit. According to Table 3.2, $h_{22}$ is calculated by connecting an open circuit across port 1 and a voltage source having voltage $V_2$ across port 2. The resulting circuit is depicted in Figure 3.6. Now, $h_{22}$ is determined from Figure 3.6(b) by calculating the port current $I_2$.

$$I_2 = \frac{V_2}{R_1 + R_2 + R_3 + \dfrac{1}{C_1 s}} + \frac{V_2}{\dfrac{1}{C_4 s}} = \frac{14s^2+12s}{7s+4} V_2 = h_{22} V_2 \tag{3.25}$$

Of course, this is the same expression as was calculated earlier from the $y$ parameters of the circuit.

Next, consider the problem of analyzing a circuit consisting of a two-port network and some external circuitry. Figure 3.7 depicts such a circuit. The currents in the resistors $R_s$ and $R_L$ are given by

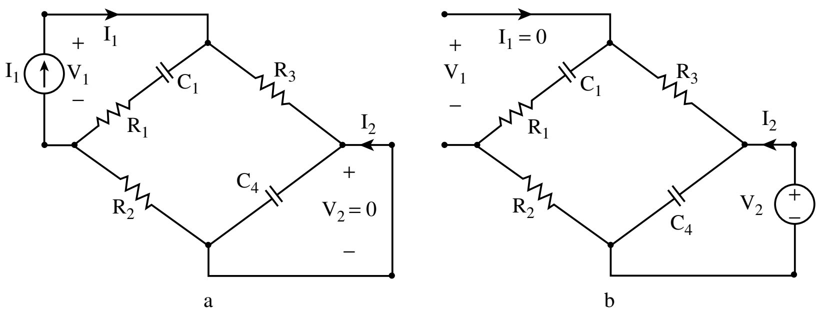



**FIGURE 3.6** The test circuits used to calculate the $h$ parameters of the example circuit.

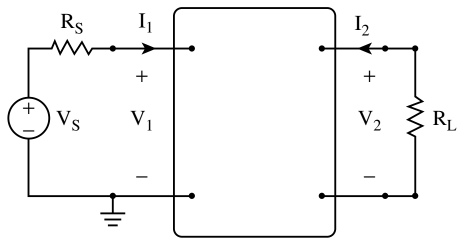



**FIGURE 3.7** An application of the two-port network.

$$I_1 = \frac{V_s - V_1}{R_s} \quad \text{and} \quad I_2 = -\frac{V_2}{R_L} \tag{3.26}$$

Suppose the two-port network used in Figure 3.7 is the circuit shown in Figure 3.4 and represented by $y$ parameters in Eq. (3.19). Combining the previous expressions for $I_1$ and $I_2$ with Eq. (3.19) yields

$$\begin{pmatrix} \dfrac{V_s}{R_s} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{14s+8}{5s+10} + \dfrac{1}{R_s} & -\dfrac{4}{5} \\ -\dfrac{4}{5} & \dfrac{10s+4}{5} + \dfrac{1}{R_L} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{3.27}$$

This equation can then be solved, e.g., using Cramer's Rule, for the transfer function

$$\begin{aligned} \frac{V_2}{V_s}(s) &= \frac{-\left(-\dfrac{4}{5}\right)\left(\dfrac{1}{R_s}\right)}{\left(\dfrac{14s+8}{5s+10} + \dfrac{1}{R_s}\right)\left(\dfrac{10s+4}{5} + \dfrac{1}{R_L}\right) - \left(-\dfrac{4}{5}\right)^2} \\ &= \frac{4(s+2)R_L}{(28s^2+24s)R_L R_s + (10s+4)(s+2)R_L + (14s+8)R_s + 5(s+2)} \end{aligned} \tag{3.28}$$

Next consider Figure 3.8. This circuit illustrates a caution regarding use of the port convention. The use of ports assumes that the currents in the terminals comprising a port are equal in magnitude and opposite in sign. This assumption is not satisfied in Figure 3.8 so port equations cannot be used to represent the four-terminal network.

Table 3.3 presents three circuit models for two port networks. These three models are based on $y$, $z$, and $h$ parameters, respectively. Such models are useful when analyzing circuits that contain subcircuits

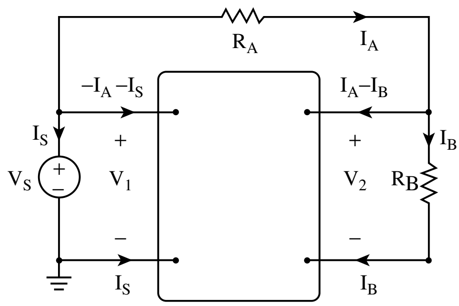



**FIGURE 3.8** An incorrect application of the two-port representation.

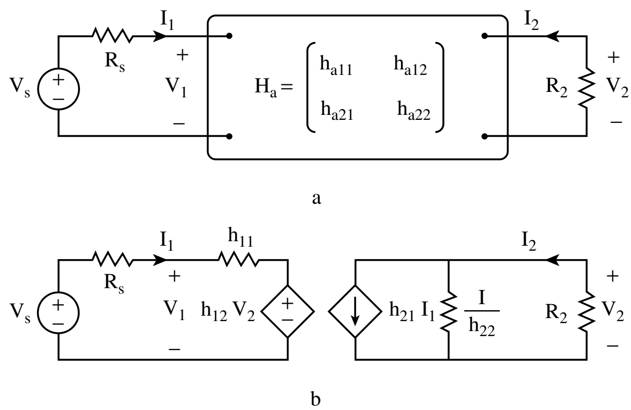



**FIGURE 3.9** Application of the circuit model associated with *H* parameters.

that are represented by port parameters. As an example, consider the circuit shown in Figure 3.9(a). This circuit contains a two port network represented by *h* parameters. In Figure 3.9(b) the two-port network has been replaced by the model corresponding to *h* parameters. Analysis of Figure 3.9(b) yields

$$\begin{aligned} V_s &= (R_s + h_{11})I_1 + h_{12}V_2 \\ V_2 &= h_{21}\frac{-R_2}{R_2 h_{22} + 1} I_1 \end{aligned} \tag{3.29}$$

After some algebra,

$$\frac{V_2}{V_s} = \frac{h_{21}R_2}{h_{12}h_{21}R_2 - (h_{11} + R_2)(h_{22}R_2 + 1)} \tag{3.30}$$

The circuits in Figure 3.9(a) and (b) are equivalent, so this is the voltage gain of the circuit in Figure 3.9(a).

H parameters are frequently used to describe bipolar transistors. Table 3.4 shows the popular methods for converting this three terminal device into a two-port network. The three configurations shown in Table 3.4 are called "common emitter," "common collector," and "common base" to indicate which terminal is common to both ports. Table 3.4 also presents the notation that is commonly used to name the *h* parameters when they are used to represent a transistor. For example, $h_{fe}$ is the forward gain when the transistor is connected in the common emitter configuration. Comparing with Table 3.3, it can be observed that $h_{fe} = h_{21}$.

**TABLE 3.3** Models of Two-Port Networks

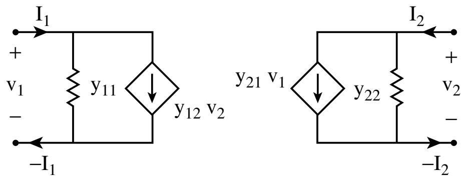
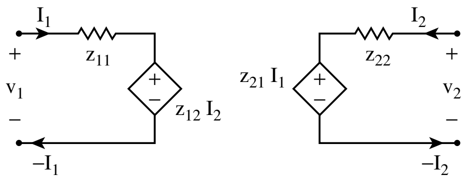
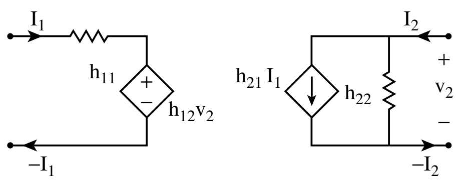
| | |
|---|---|
| $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ |   |
| $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ |   |
| $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$ |   |

Figure 3.10 depicts a popular model of a bipolar transistor. Suppose the $h$ parameters are calculated for the common emitter configuration. The result is

$$\begin{pmatrix} h_{ie} & h_{re} \\ h_{fe} & h_{oe} \end{pmatrix} = \begin{pmatrix} r_\pi & 0 \\ \beta & r_0 \end{pmatrix} \tag{3.31}$$

This calculation makes a connection between the parameters of the circuit model of the transistor and the $h$ parameters used to describe the transistor, e.g., $h_{fe} = \beta$.

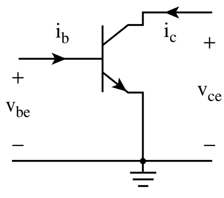
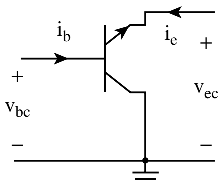
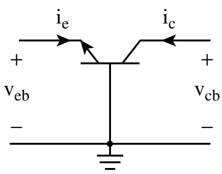
Figures 3.11 to 3.13 illustrate some common interconnections of two-port networks. In Figure 3.11, the two-port network labeled A and B are connected in cascade. As illustrated in Figure 3.11, the transmission matrix of the composite network is given by the product of the transmission matrices of the subnetworks. This simple relationship makes it convenient to use transmission parameters to represent a composite network that consists of the cascade of two subnetworks.

In Figure 3.12, the two-port networks labeled A and B are connected in parallel. As shown in Figure 3.12, the admittance matrix of the composite network is given by the sum of the admittance matrices of the subnetworks. This simple relationship makes it convenient to use admittance parameters to represent a composite network that consists of parallel subnetworks.

In Figure 3.13, the two-port network labeled A and B are connected in series. As illustrated in Figure 3.13, the impedance matrix of the composite network is given by the sum of the impedance matrices of the subnetworks. This simple relationship makes it convenient to use impedance parameters to represent a composite network that consists of series subnetworks.

Figure 3.14 is a circuit that consists of three two-port networks. Two-port parameters representing the entire network can be calculated from the two-port parameters of the subnetworks. Let $c$ denote the two-port network consisting of network $b$ connected in parallel with network $a$. Represent network $c$ with $y$ parameters by converting the $h$ parameters representing network $a$ to $y$ parameters and adding these $y$ parameters to the $y$ parameters representing network $b$.

**TABLE 3.4** Using H Parametes to Specify Bipolar Transistors

| | |
|---|---|
|   common emitter | $\begin{pmatrix} v_{be} \\ i_c \end{pmatrix} = \begin{pmatrix} h_{ie} & h_{re} \\ h_{fe} & h_{oe} \end{pmatrix} \begin{pmatrix} i_b \\ v_{ce} \end{pmatrix}$ |
|   common collector | $\begin{pmatrix} v_{bc} \\ i_e \end{pmatrix} = \begin{pmatrix} h_{ic} & h_{rc} \\ h_{fc} & h_{oc} \end{pmatrix} \begin{pmatrix} i_b \\ v_{ec} \end{pmatrix}$ |
|   common base | $\begin{pmatrix} v_{eb} \\ i_c \end{pmatrix} = \begin{pmatrix} h_{ib} & h_{rc} \\ h_{fb} & h_{oc} \end{pmatrix} \begin{pmatrix} i_b \\ v_{cb} \end{pmatrix}$ |

*i* input impedance or admittance
*o* output impedance or admittance
*f* forward gain
*r* reverse gain
*e* common emitter
*b* common base
*c* common collector

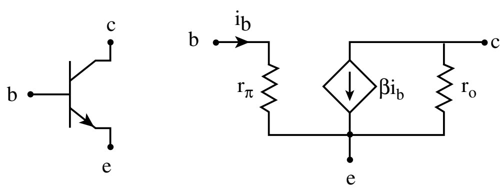
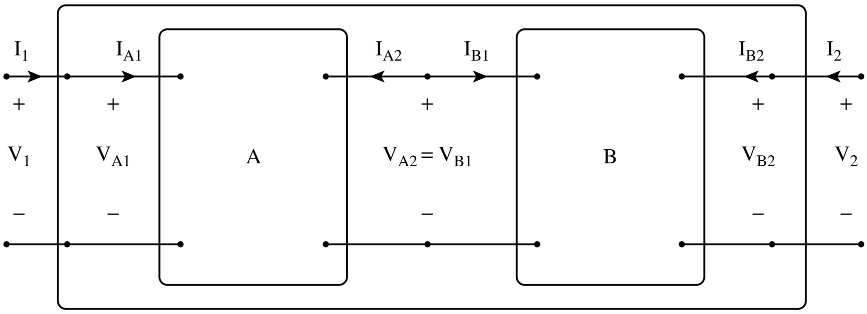



**FIGURE 3.10** The hybrid pi model of a transistor.

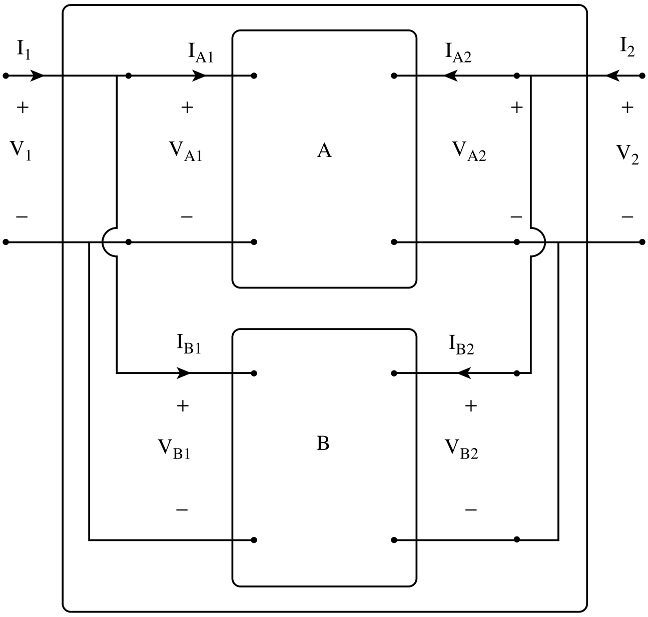
$$Y_c = \begin{pmatrix} \dfrac{1}{h_{a11}} + y_{b11} & -\dfrac{h_{a12}}{h_{a11}} + y_{b12} \\ \dfrac{h_{a21}}{h_{a11}} + y_{b21} & \dfrac{|H_a|}{h_{a11}} + y_{b22} \end{pmatrix} \triangleq \begin{pmatrix} y_{c11} & y_{c12} \\ y_{c21} & y_{c22} \end{pmatrix} \tag{3.32}$$

Next, network *c* is connected in cascade with network *d*. Represent network *d* with transmission parameters by converting the *y* parameters representing network *c* to transmission parameters and multiplying these transmissions parameters by the transmission parameters representing network *d*.




$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_a & B_a \\ C_a & D_a \end{pmatrix} \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}$$

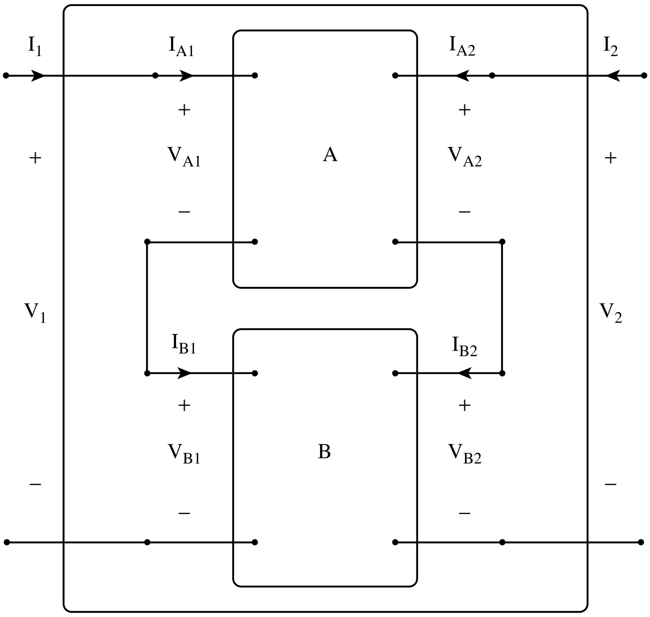
**FIGURE 3.11** Cascade connection of two-port networks.




$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} y_{A11} & y_{A12} \\ y_{A21} & y_{A22} \end{pmatrix} + \begin{pmatrix} y_{B11} & y_{B12} \\ y_{B21} & y_{B22} \end{pmatrix}$$

**FIGURE 3.12** Parallel connection of two-port networks.

$$T = \begin{pmatrix} -\dfrac{y_{c22}}{y_{c21}} & -\dfrac{1}{y_{c21}} \\ -\dfrac{|Y_c|}{y_{c21}} & -\dfrac{y_{c11}}{y_{c21}} \end{pmatrix} \begin{pmatrix} A_d & B_d \\ C_d & D_d \end{pmatrix} \triangleq \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.33}$$




$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} z_{A11} & z_{A12} \\ z_{A21} & z_{A22} \end{pmatrix} + \begin{pmatrix} z_{B11} & z_{B12} \\ z_{B21} & z_{B22} \end{pmatrix}$$

**FIGURE 3.13** Series connection of two-port networks.

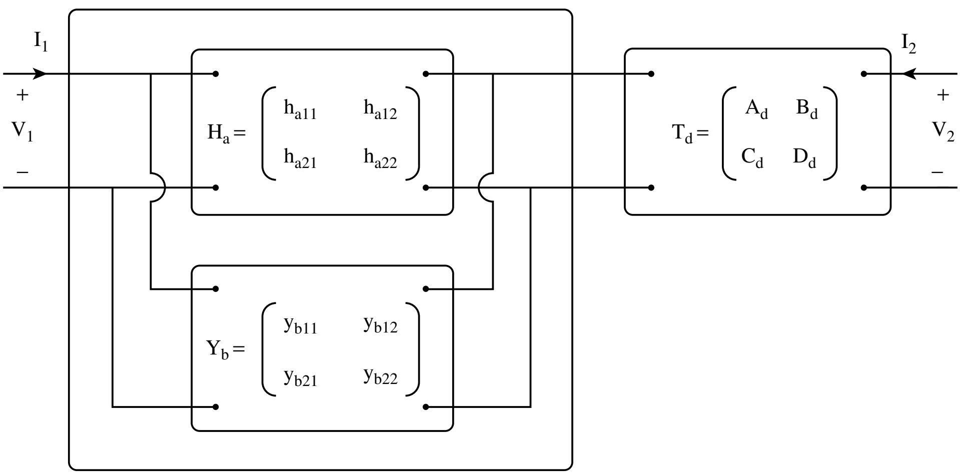



**FIGURE 3.14** A circuit consisting of three two-port networks.

Finally, the circuit in Figure 3.14 is represented by

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \tag{3.34}$$

## 3.4 Port Representations and Feedback Systems

Port representations of networks can be used to decompose a network into subnetworks. In this section, the decomposition will be done in such a way as to establish a connection between the network and a feedback system represented by the block diagram shown in Figure 3.15. Having established this connection, the theory of feedback systems can be applied to the network. Here, the problem of determining the relative stability of the network will be considered.

The theory of feedback systems [4] can be used to determine relative stability such as the one shown in Figure 3.15. The transfer function of this system is

$$T(\omega) = D(\omega) + \frac{C_1(\omega)A(\omega)C_2(\omega)}{1 + A(\omega)\beta(\omega)} \tag{3.35}$$

The phase and gain margins are parameters of a system that are used to measure relative stability. These parameters can be calculated from $A(\omega)$ and $\beta(\omega)$. To calculate the phase margin, first identify $\omega_m$ as the frequency at which

$$\begin{aligned} |A(\omega_m)||\beta(\omega_m)| = 1 &\Rightarrow |A(\omega_m)| = \frac{1}{|\beta(\omega_m)|} \\ &\Rightarrow 20\log_{10}|A(\omega_m)| = -20\log_{10}|\beta(\omega_m)| \end{aligned} \tag{3.36}$$

The phase margin is then given by

$$\begin{aligned} \phi_m &= 180^\circ + \angle\big(A(\omega_m)\beta(\omega_m)\big) \\ &= 180^\circ + \big(\angle A(\omega_m) + \angle\beta(\omega_m)\big) \end{aligned} \tag{3.37}$$

The gain margin is calculated similarly. First, identify $\omega_p$ as the frequency at which

$$180^\circ = \angle\big(A(\omega_p)\beta(\omega_p)\big) \tag{3.38}$$

The gain margin is given by

$$GM = \frac{1}{|A(\omega_p)||\beta(\omega_p)|} \tag{3.39}$$

Next, consider an active circuit which can be represented as shown in Figure 3.16. For convenience, it is assumed that the input and output of the circuit are both node voltages. An amplifier has been selected and separated from the rest of the circuit. The rest of the circuit has been denoted as *N*.

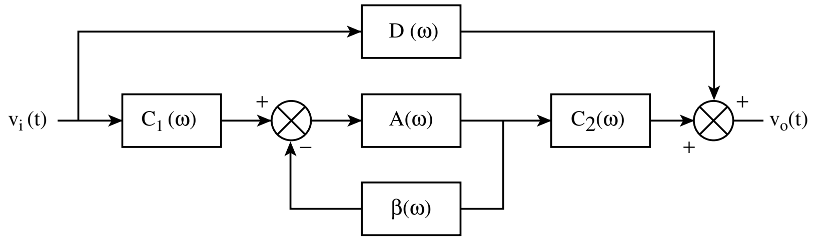



**FIGURE 3.15** A feedback system.

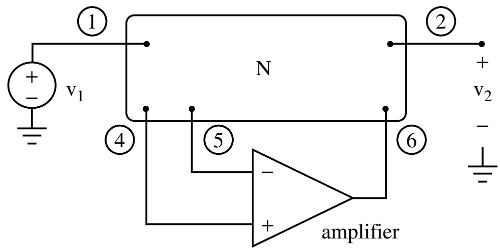



**FIGURE 3.16** Identifying the network $N$.

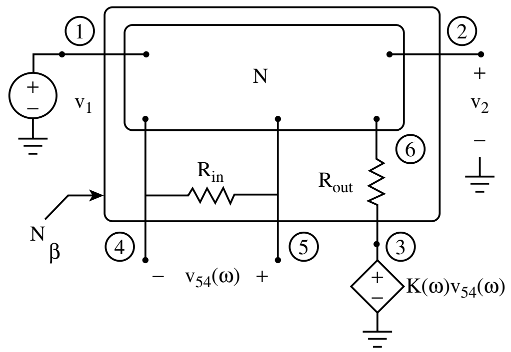



**FIGURE 3.17** Identifying the Beta Network, $N_\beta$.

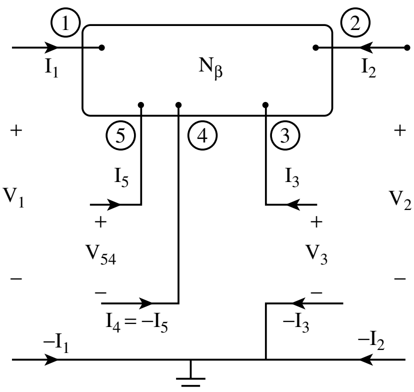



**FIGURE 3.18** Identifying the ports of the Beta Network.

Suppose that the amplifier is a voltage controlled voltage source (VCVS), e.g., an inverting or a noninverting amplifier or an op amp. Replacing the VCVS by a simple model yields the circuit shown in Figure 3.17. (The VCVS model accounts for input and output resistance and frequency dependent gain.) Figure 3.17 shows how to identify the network $N_\beta$ which consists of the network $N$ from Figure 3.16 together with the input and output resistances of the VCVS. The network $N_\beta$ has been called the "Beta Network" [5].

Figure 3.18 illustrates a way of grouping the terminals of the five-terminal network $N_\beta$ to obtain a four-port network. This four-port network can be represented by the hybrid equation

$$\begin{pmatrix} I_1 \\ V_2 \\ I_3 \\ V_{54} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} \begin{pmatrix} V_1 \\ I_2 \\ V_3 \\ I_5 \end{pmatrix} \tag{3.40}$$

Because $I_1$ and $I_3$ are not of interest, the first and third rows of this equation can be set aside. Then setting $I_2$ and $I_5$ equal to zero yields

$$\begin{pmatrix} V_2 \\ V_{54} \end{pmatrix} = \begin{pmatrix} H_{21} & H_{23} \\ H_{41} & H_{43} \end{pmatrix} \begin{pmatrix} V_1 \\ V_3 \end{pmatrix} \tag{3.41}$$

where, for example

$$H_{43}(\omega) = \frac{V_{54}(\omega)}{V_3(\omega)} \quad \text{when} \quad V_1(\omega) = 0 \tag{3.42}$$

and $H_{21}(\omega)$, $H_{23}(\omega)$ and $H_{41}(\omega)$ are defined similarly.

The amplifier model requires that

$$V_3(\omega) = K(\omega) V_{54}(\omega) \tag{3.43}$$

Combining these equations yields

$$T(\omega) = \frac{V_2(\omega)}{V_1(\omega)} = H_{21}(\omega) + \frac{K(\omega) H_{23}(\omega) H_{41}(\omega)}{1 - K(\omega) H_{43}(\omega)} \tag{3.44}$$

Comparing this equation with the transfer function of the feedback system yields

$$\begin{aligned} A(\omega) &= -K(\omega) \\ \beta(\omega) &= H_{43}(\omega) \\ C_1(\omega) &= H_{23}(\omega) \\ C_2(\omega) &= H_{41}(\omega) \\ D(\omega) &= H_{21}(\omega) \end{aligned} \tag{3.45}$$

These equations establish a correspondence between the feedback system in Figure 3.15 and the active circuit in Figure 3.16. This correspondence can be used to identify $A(s)$ and $\beta(s)$ corresponding to a particular circuit. Once $A(s)$ and $\beta(s)$ are known, the phase or gain margin can be calculated.

Figure 3.19 shows a Tow–Thomas bandpass biquad [3]. When the op amps are ideal devices the transfer function of this circuit is

$$T(s) = \frac{V_{in}(s)}{V_{out}(s)} = \frac{-\frac{1}{KRC} s}{s^2 + \frac{s}{QRC} + \frac{1}{R^2C^2}} \tag{3.46}$$

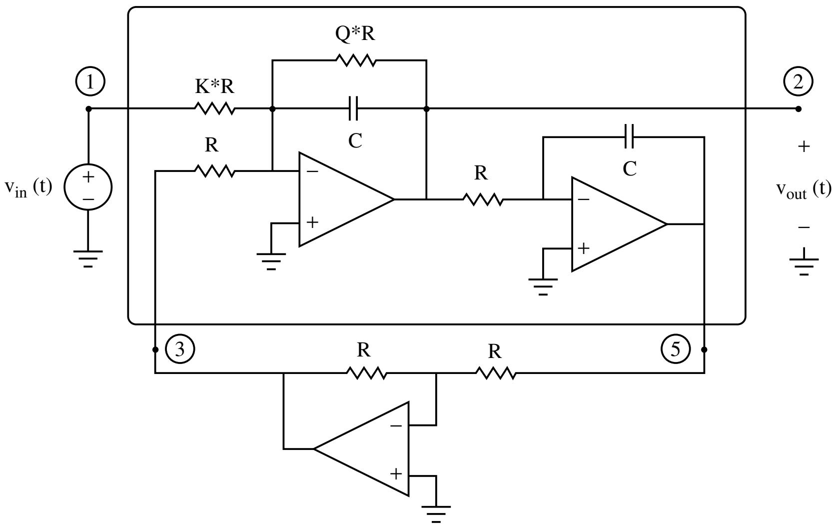



**FIGURE 3.19** The Tow–Thomas biquad.

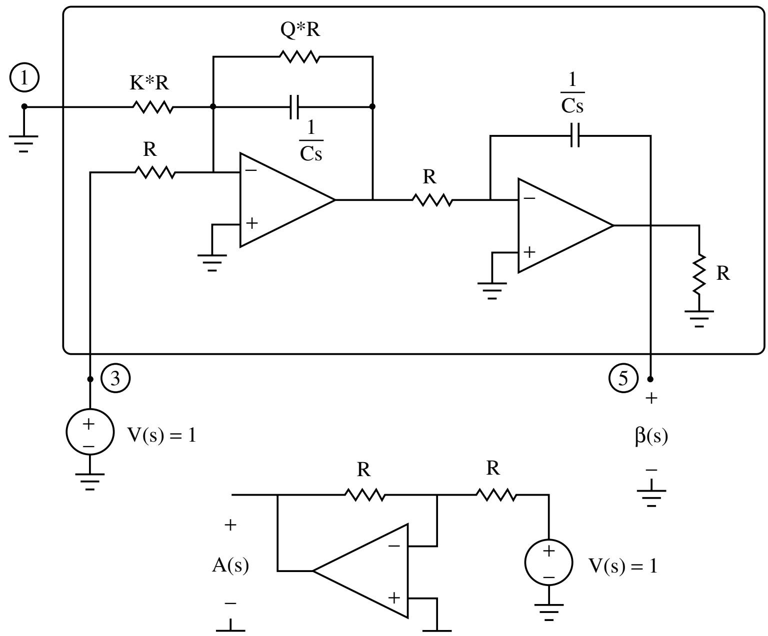



**FIGURE 3.20** Calculating $A(s)$ and $\beta(s)$ for the Tow–Thomas biquad.

Figure 3.20 illustrates circuits that can be used to identify $A(s)$ and $\beta(s)$.

$$A(s) = -1$$
$$\beta(s) = \frac{Q}{QR^2C^2s^2 + RCs} \tag{3.47}$$

Next,

$$1 = \left|A(\omega_m)\right|\left|\beta(\omega_m)\right| \Rightarrow (RC\omega_m)^2 = \frac{\frac{1}{Q^2} \pm \sqrt{\frac{1}{Q^4} + 4}}{2} \tag{3.48}$$

The phase margin is given by

$$\begin{aligned}\phi_m &= 180^\circ + \angle A(\omega_m) + \angle \beta(\omega_m) \\ &= 180^\circ + 180^\circ - \tan^{-1}\frac{-1}{QRC\omega_m} \\ &= \tan^{-1}\frac{1}{QRC\omega_m}\end{aligned} \tag{3.49}$$

When Q is large,

$$\omega_m \approx \frac{1}{RC} \Rightarrow \phi_m \approx \tan^{-1}\frac{1}{Q} \tag{3.50}$$

When the op amps are not ideal, it is not practical to calculate the phase margin by hand. With computer-aided analysis, accurate amplifier models, such as macromodels, can easily be incorporated into this analysis [5].

## 3.5 Conclusions

It is frequently useful to decompose a large circuit into subcircuits. These subcircuits are connected together at ports and terminals. Port and terminal parameters describe how the subcircuits interact with other subcircuits but do not describe the inner workings of the subcircuit itself.

This section has presented procedures for determining port and terminal parameters and for analyzing networks consisting of subcircuits which are represented by port or terminal parameters. Port equations were used to establish a connection between electronic circuits and feedback systems.

### References

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[3] L. P. Huelsman and P. E. Allen, *Theory and Design of Active Filters*, New York: McGraw-Hill, 1980.
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