4

Signal Flow Graphs in Filter Analysis and Synthesis

Pen-Min Lin
Purdue University

4.1 Formulation of Signal Flow Graphs for Linear Networks

Any lumped network obeys three basic laws: Kirchhoff's voltage law (KVL), Kirchhoff's current law (KCL), and the elements' laws (branch characteristics). For filter applications, we write the frequency-domain instead of the time-domain network equations. Three general methods for writing network equations are described in Chapter 2.1. They are the node equations, the loop equations, and the hybrid equations. This section outlines another method, the signal flow graph (SFG) method of characterizing a linear network.

Consider first the construction of signal flow graphs for linear networks without controlled sources. For all practical networks, the independent voltage sources (E) contain no loops, and the independent current sources (J) contain no cutsets. Under these conditions, it is always possible to select a tree T, such that all voltage sources are included in the tree and all current sources are included in the co-tree. The network branches are divided into four sets (each set may be empty) indicated by subscripts as follows:

E: independent voltage sources
J: independent current sources
Z: passive branches in the tree, characterized by impedances
Y: passive branches in the co-tree, characterized by admittances

A step-by-step procedure for constructing an SFG is given below.

Procedure 1 (for linear networks without controlled sources)

Step 1. Apply KVL to express each V_Y (voltage of a passive branch in the co-tree) in terms of V_E and V_Z.
Step 2. Apply KCL to express each I_Z (current of a passive branch in the tree) in terms of I_J and I_Y.
Step 3. For each passive tree branch, consider its voltage as the product of impedance and current, i.e., $V_Z = Z_Z I_Z$.
Step 4. For each passive co-tree branch, consider its current as the product of admittance and voltage, i.e., $I_Y = Y_Y V_Y$.

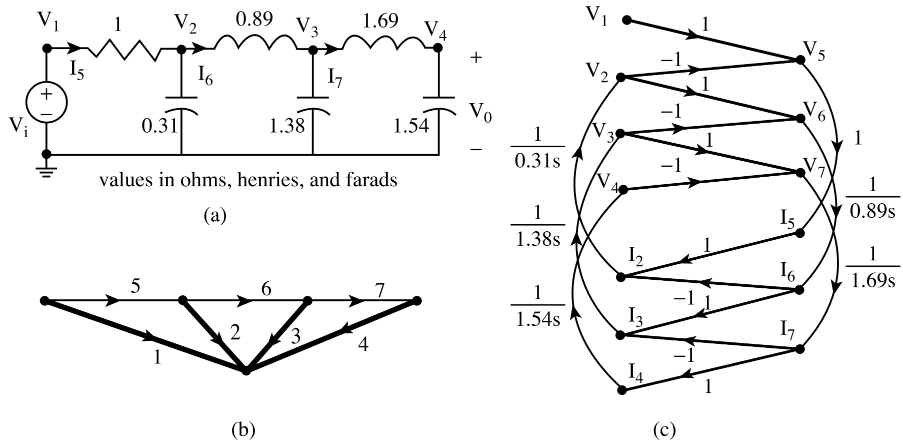


FIGURE 4.1 (a) A low-pass filter network. (b) Directed graph for the network and a chosen tree. (c) SFG based on the chosen tree.

Example 1. Construct a signal flow graph for the low-pass filter network shown in Figure 4.1(a), and use Mason's formula to find the voltage gain function $H(s) = V_o(s)/V_i(s)$.

Solution. The graph associated with the network is shown in Figure 4.1(b) in which the branch numbers and reference directions (passive sign convention) have been assigned. The complexity of the SFG depends on the choice of the tree. In the case of a ladder network, a good tree to use is a star tree which has all tree branches connected to a common node. For the present network, we choose the tree to be $T = \{1, 2, 3, 4\}$, shown in heavy lines in Figure 4.1(b).

Step 1 yields: $V_5 = V_i - V_2, V_6 = V_2 - V_3, V_7 = V_3 - V_4$

Step 2 yields: $I_2 = I_5 - I_6, I_3 = I_6 - I_7, I_4 = I_7$

Step 3 yields:

$$V_2 = \frac{1}{0.31s} I_2$$

$$V_3 = \frac{1}{1.38s} I_3$$

$$V_4 = \frac{1}{1.54s} I_4$$

Step 4 yields:

$$I_5 = V_5$$

$$I_6 = \frac{1}{0.89s} V_6$$

$$I_7 = \frac{1}{1.69s} V_7$$

The SFG of Figure 4.1(c) displays all the preceding relationships.

Applying Mason's gain formula to the SFG of Figure 4.1(c), we find

$$H(s) = \frac{V_o}{V_i} = \frac{V_4}{V_1} = \frac{1}{s^5 + 3.24s^4 + 5.24s^3 + 5.24s^2 + 3.24s + 1}$$

Next, consider linear networks containing controlled sources. All four types of controlled sources may be present. Our strategy is to utilize procedure 1 described previously with some pre-analysis manipulations. The following is a step-by-step procedure.

Procedure 2 (for linear networks containing controlled sources)

Step 1. *Temporarily* replace each controlled voltage source by an independent voltage source, and each controlled current source by an independent current source, while retaining their original reference directions. The resultant network has no controlled sources.
Step 2. Construct the SFG for the network obtained in step 1 using procedure 1.
Step 3. Express the desired outputs and all controlling variables, if they are not present in the SFG, in terms of the quantities already present in the SFG.
Step 4. Reinstate the constraints of all controlled sources.

Example 2. Construct an SFG for the amplifier circuit depicted in Figure 4.2(a).

Solution. We first replace the controlled voltage source μV_g by an independent voltage source V_x. A tree is chosen to be (V_s, R_a, V_x). The result of step 1 of procedure 2 is depicted in Figure 4.2(b) where dashed lines indicate co-tree branches.

For the links R_b and R_c, we have $I_b = G_b V_b = G_b(V_s - V_a + V_x)$, and $I_c = G_c V_c = -G_c V_x$. For the tree branch R_a we have $V_a = R_a I_a = R_b I_b$. The result of step 2 of procedure 2 is depicted in Figure 4.2(c). Note that the simple relationships $V_b = (V_s - V_a + V_x)$, $V_c = -V_x$ and $I_a = I_b$ have been used to eliminate the variables V_b, V_c and I_a. As a result, these variables do not appear in Figure 4.2(c).

The desired output is $V_o = -V_x$ and the controlling voltage is $V_g = V_s - V_a$. After expressing these relationships in the SFG, step 3 of procedure 2 results in Figure 4.2(d).

Finally, we reinstate the constraint of the controlled source, namely, $V_x = \mu V_g$. The result of step 4 of procedure 2, in Figure 4.2(e), is the desired SFG.

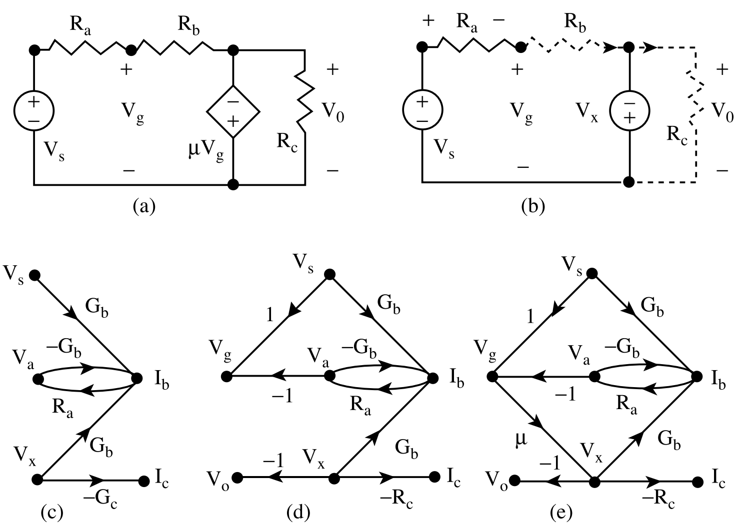


FIGURE 4.2 (a) A linear active network. (b) Result of step 1, procedure 2. (c) Result of step 2, procedure 2. (d) Result of step 3, procedure 2. (e) The desired SFG.

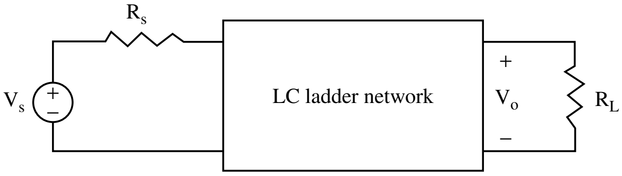


FIGURE 4.3 A doubly terminated passive filter.

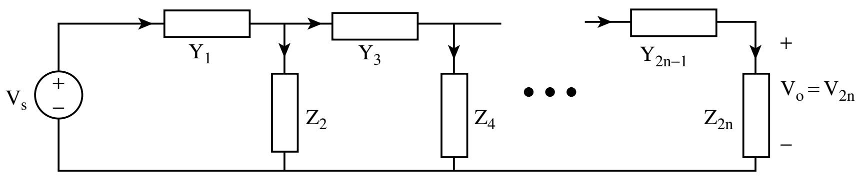


FIGURE 4.4 A general ladder network.

4.2 Synthesis of Active Filters Based on Signal Flow Graph Associated with a Passive Filter Circuit

The preceding section demonstrates that the equations governing a linear network can be described by an SFG in which the branch weights (or transmittances) are either real constants or simple expressions of the form K_s or K/s. All the cause–effect relationships displayed in such an SFG can, in turn, be implemented with resistors, capacitors, and ideal operational amplifiers. The inductors are *not* needed in the implementation. Whatever frequency response prevailing in the original linear circuit appears exactly in the RC-op-amp circuit.

In active filter synthesis, the method described in Section 4.1 is applied to a passive filter in the form of an LC (inductor-capacitor) ladder network terminated in a resistance at both ends as illustrated in Figure 4.3. The reason is that this type of filter, with $R_s = R_L$, has been proved to have the best sensitivity property [1, p. 196]. By this, we mean that the frequency response is least sensitive with respect to the changes in element values, when compared to other types of filter circuits. Because magnitude scaling (i.e., multiplying all impedances in the network by a factor K_m) does not affect the voltage gain function, we always normalize the prototype passive filter network so that the source resistance becomes 1 Ω. The advantage of this normalization will become evident in several examples given in this section.

The SFG illustrated in Figure 4.1(c) has many branches crossing each other. For a ladder network, with a proper choice of the tree and a rearrangement of the SFG nodes, all crossings can be eliminated. To achieve this, we first label a general ladder network as shown in Figure 4.4.

The following conventions are used in the labels of Figure 4.4:

1. All series branches are numbered odd and characterized by their admittances.
2. All shunt branches are numbered even and characterized by their impedances.
3. A single arrow is used to indicate the reference directions of both the voltage and the current of each network branch. Passive sign convention is used.
4. If the LC ladder in Figure 4.4 has a series element at the source end, then Y_1 represents that element in series with R_s.
5. If the LC ladder in Figure 4.4 has a shunt element at the load end, then Z_{2n} represents that element in parallel with R_L.

For constructing the SFG, choose a tree to consist of the voltage source and all shunt branches. The SFG for the circuit may be constructed using procedure 1 of Section 4.1. First, list the equations obtained in each step.

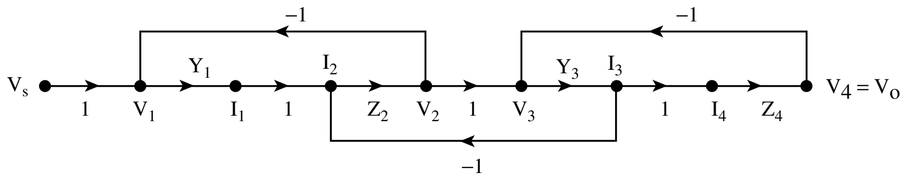


FIGURE 4.5 SFG for a 4-element ladder network.

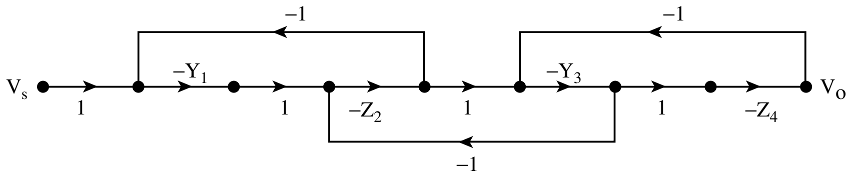


FIGURE 4.6 Inverting integrators are used in this modified SFG.

Step 1. $V_1 = V_s - V_2,\ \ V_3 = V_2 - V_4,\ \ldots, V_{2n-1} = V_{2n-2} - V_{2n}$

Step 2. $I_2 = I_1 - I_3,\ \ I_4 = I_3 - I_5, \ldots, I_{2n} = I_{2n-1}$

Step 3. $V_2 = Z_2 I_2,\ \ V_4 = Z_4 I_4, \ldots, V_{2n} = Z_{2n} I_{2n}$

Step 4. $I_1 = Y_1 V_1,\ \ I_3 = Y_3 V_3, \ldots, I_{2n-1} = Y_{2n-1} V_{2n-1}$

These relationships are represented by the SFG in Figure 4.5 for the case of a four-element ladder network. Note that the SFG graph nodes have been arranged in such a way that there are no branch crossings. The pattern displayed in this SFG suggests the children's game of *leapfrog*. Consequently, an active filter synthesis based on the SFG of Figure 4.5 is called a leapfrog realization. The transmittance of each SFG branch indicates the type of mathematical operation performed. For example, $1/s$ means integration and is implemented by an op amp integrator. Likewise, $1/(s + a)$ is implemented by a lossy op amp integrator. It is well known that inverting integrators and inverting summers can be designed with singled-ended op amps (i.e., the noninverting input terminal of each op amp is grounded), [2–5]. Noninverting integrators and noninverting summers can also be designed, but require differential-input op amps and more complex circuitry. Therefore, there is an advantage in using the inverting types. To this end, we multiply all Z's and Y's in Figure 4.5 by –1, with the result shown in Figure 4.6. Note that in Figure 4.6 we have removed the labels of internal SFG nodes because they are of no consequence in determining the transfer function. The transfer function V_o/V_s is the same for both Figure 4.5 and Figure 4.6. This is quite obvious from Mason's gain formula, as all path weights and loop weights are not affected by the modification. A branch transmittance of –1 indicates an inverting amplifier. In the interest of reducing the total number of op amps used, we want to reduce the number of branches in the SFG that have weight –1. This can be achieved by inserting branches weighted –1 in some strategic places. Each insertion will lead to the change of the signs of one or two feedback branches. The rules are (a) inserting a branch weighted –1 in a forward path segment shared by two feedback loops changes the signs of the two feedback branch weights; (b) inserting a branch weighted –1 in a forward path segment belonging to one feedback loop only changes the signs of that feedback branch weight. Figure 4.6 is modified this way and the result is shown in Figure 4.7. The inserted branches are shown in heavy lines.

Comparing Figure 4.6 with Figure 4.7, we see that there is no change in path weights and loop weights. Therefore, Mason's gain formula assures that both SFG have the same transfer function. For a six-element ladder network, three branches weighted –1 must be inserted. This leads to a sign change of the single forward path weight in the SFG, and the output node variable now becomes $-V_o$. For filter applications

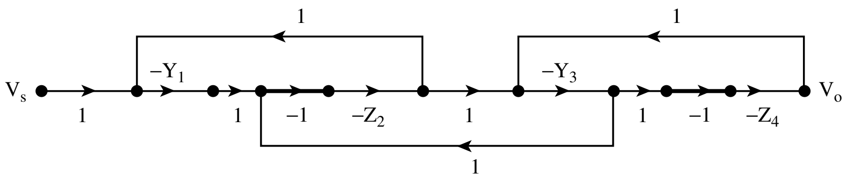
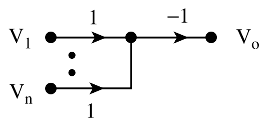
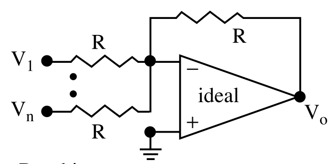
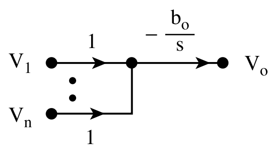
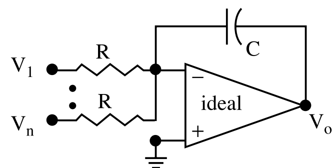
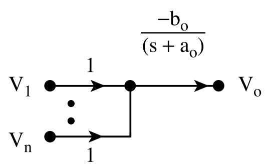
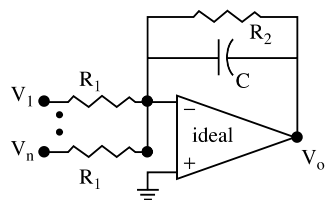


FIGURE 4.7 Modification to reduce the number of inverting amplifiers.

TABLE 4.1 Component Op Amp Circuits for Synthesizing Active Low-Pass Filters by the Leapfrog Technique

(1)	 $V_o = -(V_1 + \ldots + V_n)$	Inverting summer  R: arbitrary
(2)	 $V_o = -\frac{b_o}{s}(V_1 + \ldots + V_n)$	Inverting summing integrator  C: arbitrary, $R = \frac{1}{b_o C}$
(3)	 $V_o = \frac{-b_o}{(s + a_o)} \quad (V_1 + \ldots + V_n)$	Inverting summing lossy integrator  C: arbitrary, $R_1 = \frac{1}{b_o C}$, $R_2 = \frac{1}{a_o C}$,

Note: Each component RC-op-amp circuit in the right column may be magnitude-scaled by an arbitrary factor.

this change of sign in the transfer function is acceptable as we are concerned mainly with the magnitude response.

An implementation of the SFG of Figure 4.7 may be accomplished easily by referring to Table 4.1 and picking the component op amp circuits for realizing the SFG transmittances −1, $-Y_1$, $-Z_2$, etc. Figure 4.7 dictates how these component op amp circuits are interconnected to produce the desired voltage gain function. An example will illustrate the procedure.

Example 3. Figure 4.8 shows a normalized Butterworth fourth-order, low-pass filter, where the 1-Ω source resistance has been included in Y_1, and the 1-Ω load resistance included in Z_4.

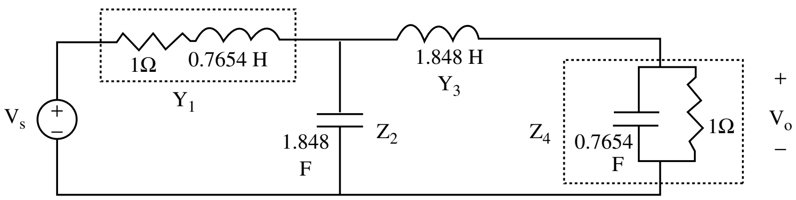


FIGURE 4.8 A fourth-order, Butterworth low-pass filter.

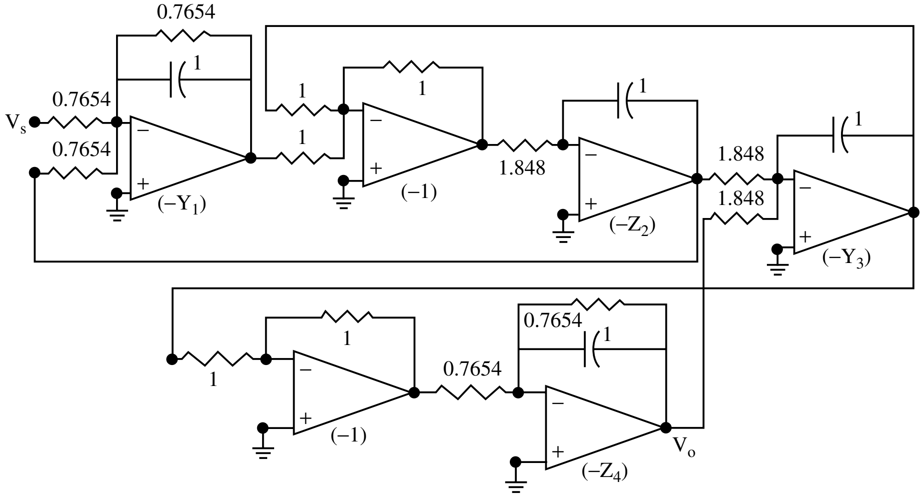


FIGURE 4.9 Leapfrog realization of passive filter of Figure 4.8. For the normalized case of $\omega_{3dB} = 1$ r/sec, values are in Ω and F. For a practical case of $\omega_{3dB} = 10^6$ rad/s, values are in kΩ and nF.

The leapfrog-type SFG for this circuit, after suitable modifications, is shown in Figure 4.7, where

$$-Y_1 = -\frac{1}{0.7654s+1}$$

$$-Z_2 = -\frac{1}{1.848s}$$

$$-Y_3 = -\frac{1}{1.848s}$$

$$-Z_4 = -\frac{1}{0.7654s+1}$$

The SFG branch transmittance $-Z_2$ and $-Y_3$ are realized using item (2) of Table 4.1, while $-Y_1$ and $-Z_4$ use item (3). The two SFG branches with weight –1 in Figure 4.7 require item (1). The SFG branches with weight 1 merely indicate how to feed the inputs to each component network. No additional op amps are needed for such SFG branches. Thus, a total of six op amps are required. The interconnection of these component circuits is described by Figure 4.7. The complete circuit is shown in Figure 4.9. One-farad capacitances have been used in the circuit. Recall that the original passive low-pass filter has a 3-dB frequency of 1 rad/s. By suitable magnitude scaling and frequency scaling, all element values in the active filter of Figure 4.9 can be made practical. For example, if the 3-dB frequency is changed to 10^6 rad/s, then the capacitances in Figure 4.9 are divided by 10^6. We may arbitrarily magnitude scale the resultant circuit by a factor of 10^3. Then, all resistances are multiplied by 10^3 and all capacitances are further divided by 10^3. The final circuit is still the one shown in Figure 4.9, but with resistances in kΩ and capacitance in nF. The parenthetical quantity beside each op amp indicates the type of transfer function it produces.

If a doubly terminated passive filter has a shunt reactance at the source end and a series reactance the load end, then its dual network has a series reactance at the source end and a shunt reactance at the load end. The voltage gain functions of the original network and its dual differ at most by a multiplying constant. We can apply the method to the dual network.

For doubly terminated Butterworth and Chebyshev low-pass filters of odd orders, the passive filter either has series reactances or shunt reactances at both ends. The next example shows the additional SFG manipulations needed to construct the RC-op-amp circuit.

Example 4. Obtain a leapfrog realization of the third-order Butterworth low-pass filter shown in Figure 4.10(a).

Solution. The network is again a four-element ladder network with a modified SFG in terms of the series admittances and shunt impedances as depicted in Figure 4.6. Note that the 1-Ω source resistance alone constitutes the element Y_1. Inserting a branch weighted −1 in front of $-Y_3$ changes the weights of two feedback branches from −1 to 1, and the output from V_o to $-V_o$. Figure 4.10(b) gives the result. Next, apply the node absorption rule to remove nodes V_A and V_B in Figure 4.10b. The result is Figure 4.10(c).

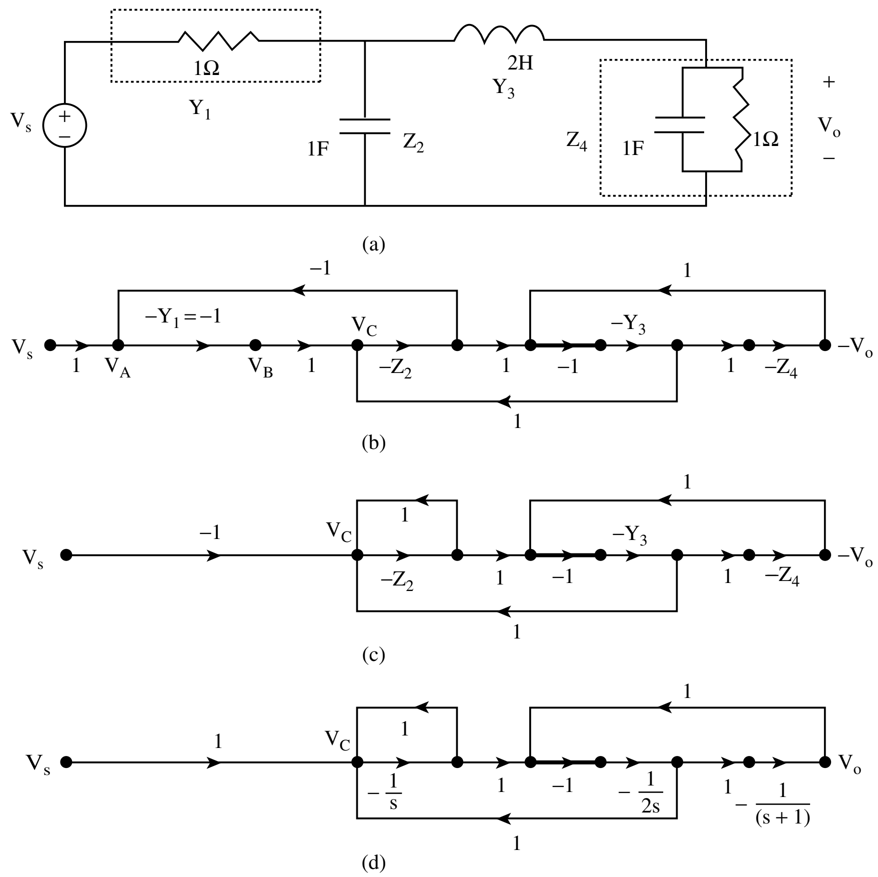


FIGURE 4.10 Leapfrog realization of a third-order, Butterworth low-pass filter. (a) The passive prototype. (b) Leapfrog SFG simulation. (c) Absorption of SFG nodes. (d) Final SFG for active filter realization.

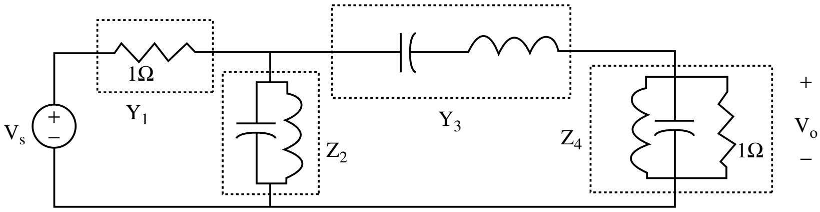


FIGURE 4.11 A bandpass passive filter derived from the circuit of Figure 4.10(a).

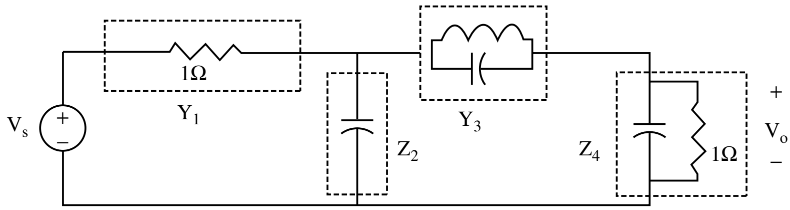


FIGURE 4.12 Network configuration of a doubly terminated filter having a third-order, elliptic or inverse Chebyshev low-pass response.

Finally, we recognize that the left-most branch weight –1 is not contained in any loop weights, and appears in the single forward path weight. Therefore, if this branch weight is changed from –1 to 1, the output will be changed from $-V_o$ to V_o. When this change is made, and all specific branch weights are used, the final SFG is given in Figure 4.10(d). The circuit implementation is now a simple matter of picking component networks from Table 4.1 and connecting them as in Figure 4.10(d). A total of four op amps are required, one each for the branch transmittance $-1/s$, $-1/(2s)$, $-1/(s+1)$, and –1.

Passive bandpass filters may be derived from low-pass filters using the frequency transformation technique described in Chapter 72. The configuration of a bandpass filter derived from the third-order Butterworth filter of Figure 4.10(a) is given in Figure 4.11.

The impedance and admittance functions Z_2, Y_3, and Z_4 are of the form

$$\frac{s}{a_2 s^2 + a_1 s + a_0}$$

The SFG thus contains quadratic branch transmittances. Several single op amp realizations of the quadratic transmittances are discussed in Chapter 82, while some multiple op amp realizations are presented in the next subsection. The interconnection of the component networks, however, is completely specified by an SFG similar to Figure 4.7 or Figure 4.10(d). Complete design examples of this type of bandpass active filter may be found in many books [2–5].

The previous example shows the application of the leapfrog technique to low-pass and bandpass filters of the Butterworth or Chebyshev types. The technique, when applied to a low-pass filter having an elliptic response or an inverse Chebyshev response will require the use of some differentiators. The configuration of a third order low-pass elliptic filter or inverse Chebyshev filter is depicted in Figure 4.12. Notice that Y_3 has the expression

$$Y_3 = a_2 s + \frac{a_0}{s} = \frac{a_2 s^2 + a_0}{s}$$

The term $a_2 s$ in the voltage gain function of the component network clearly indicates the need of a differentiator. An example of such a design may be found in [1, pp. 382–385].

As a final point in the leapfrog technique, consider the problem of impedance normalization. In all the previous examples, the passive prototype filter has equal terminations and has been magnitude-scaled so that $R_s = R_L = 1$. Situations occur where the passive filter has unequal terminations. For example, the passive filter may have $R_s = 100\ \Omega$ and $R_L = 400\ \Omega$ in a four-element ladder network in Figure 4.8. Three possibilities will be considered.

(1) No impedance normalization is done on the passive filter. Then,

$$-Y_1 = -\frac{1}{L_1 s + 100}$$

$$-Z_4 = -\frac{1}{C_4 s + \frac{1}{400}}$$

From Table 4.1, the lossy integrator realizing $-Y_1$ has a resistance ratio of 100, and the resistance ratio for the $-Z_4$ circuit is 400. Such a large ratio is undesirable.

(2) An impedance normalization is done with $R_o = 100\ \Omega$ so that R_s becomes 1 and R_L becomes 4. Then

$$-Y_1 = -\frac{1}{L_1 s + 1}$$

$$-Z_4 = -\frac{1}{C_4 s + \frac{1}{4}}$$

The resistance ratio in the lossy integrator now becomes 1 for the $-Y_1$ circuit, and 4 for the $-Z_4$ circuit — an obvious improvement over the non-normalized case.

(3) An impedance normalization is done with $R_o = \sqrt{R_s R_L} = 200$. Then $R_s = 0.5$, $R_L = 2$, and

$$-Y_1 = -\frac{1}{L_1 s + 0.5}$$

$$-Z_4 = -\frac{1}{C_4 s + 0.5}$$

The resistance ratio in the lossy integrator is now 2 for both the $-Y_1$ circuit and the $-Z_4$ circuit — a further improvement over case (2) using $R_o = R_s$.

The conclusion is that, in the interest of reducing the spread of resistance values, the best choice of R_o for normalizing the passive filter is $R_o = \sqrt{R_s R_L}$. For the case of equal terminations, this choice leads to $R_s = R_L = 1$.

Instead of starting with a normalized passive filter, one can also construct a leapfrog-type SFG based on the unnormalized passive filter. For a four-element ladder network, the result is given in Figure 4.7. We now perform the following SFG manipulation, which has the same effect as the impedance normalization of the passive filter: Select a normalization resistance, R_o and divide all Z's in the SFG by R_o, and multiply all Y's by R_o. The resultant SFG is given in Figure 4.13.

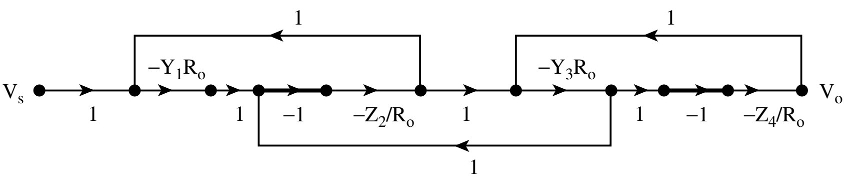


FIGURE 4.13 Result of normalization of the SFG of Figure 4.7.

It is easy to see that the SFG in both Figures 4.7 and 4.13 have the same loop weights and single forward path weight. Therefore, the voltage gain function remains unchanged with the normalization process. One advantage of using the normalized SFG is that the branch transmittances Y_kR_o and Z_k/R_o are dimensionless, and truly represent voltage gain function of component op amp circuits [2, p. 288].

4.3 Synthesis of Active Filters Based on Signal Flow Graph Associated with a Filter Transfer Function

The preceding section describes one application of the SFG in the synthesis of active filters. The starting point is a passive filter in the form of a doubly terminated LC ladder network. In this section, we describe another way of using the SFG technique to synthesize an active filter. The starting point in this case is a filter transfer function instead of a passive network.

Let the transfer voltage ratio function of a filter be

$$\frac{V_o}{V_i} = H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}, \ m \le n \tag{4.1}$$

By properly selecting the coefficient a's and b's, all types of filter characteristics can be obtained: low-pass, high-pass, bandpass, band elimination, and all-pass. We assume that these coefficients have been determined. Our problem is how to realize the transfer function using SFG theory and RC-op-amp circuits.

For the present application, we impose two constraints on the signal flow graph:

1. No second-order or higher-order loops are present. In other words, all loops in the SFG touch each other.
2. Every forward path from the source node to the output node touches all loops.

For such a special kind of SFG, Mason's gain formula reduces to

$$\frac{V_o}{V_i} = H(s) = \frac{\sum P_k}{1 - (L_1 + L_2 + \ldots + L_n)} \tag{4.2}$$

where L_n is the nth loop weight, P_k is the kth forward path weight, and summations are over all forward paths and all loops. Our strategy is to manipulate Eq. (4.1) into the form of Eq. (4.2), and then construct an SFG to have the desired loops and paths, meeting constraints (1) and (2). Integrators are preferred over differentiators in actual circuit implementation, therefore, we want $1/s$ instead of s to appear as the SFG branch transmittances. This suggests the division of both the numerator and denominator of Eq. (4.1) by s^n, the highest degree term in the denominator.

The result is

$$\frac{V_o}{V_i} = H(s) = \frac{b_m\left(\frac{1}{s}\right)^{n-m} + b_{m-1}s^{n-m+1} + \ldots + b_1\left(\frac{1}{s}\right)^{n-1} + b_o\left(\frac{1}{s}\right)^{n}}{1 + a_{n-1}\left(\frac{1}{s}\right) + \ldots + a_1\left(\frac{1}{s}\right)^{n-1} + a_0\left(\frac{1}{s}\right)^{n}}, \ m \le n \tag{4.3}$$

Comparing Eq. (4.3) with Eq. (4.2), we can identify the loop weights

$$\begin{aligned} L_1 &= -a_{n-1}(1/s) \\ L_2 &= -a_{n-2}(1/s)^2 \\ &\cdots \\ L_n &= -a_o(1/s)^n \end{aligned} \tag{4.4}$$

and the forward path weights

$$b_m\left(\frac{1}{s}\right)^{n-m}, \quad b_{m-1}\left(\frac{1}{s}\right)^{n-m+1}, \quad \ldots b_1\left(\frac{1}{s}\right)^{n-1}, \quad b_o\left(\frac{1}{s}\right)^{n} \tag{4.5}$$

Many SFGs may be constructed to have such loop and path weights, and the touching properties stated previously in (1) and (2). Two simple ones are given in Figure 4.14(a) and (b) for the case $n = m = 3$. The extension to higher-order transfer functions is obvious. In control theory, the system represented by Figure 4.14(a) is said to be of the controllable canonical form, and Figure 4.14(b) the observable canonical form. In a filter application, we need not be concerned about the controllability and observability of the system. The terms are used here merely for the purpose of circuit identification. Our major concern here is how to implement the SFG by an RC-op-amp circuit.

An SFG branch having transmittance $1/s$ indicates an integrator. If the terminating node of the $1/s$ branch has no other incoming branches [as in Figure 4.14(a)], then that node variable represents the output of the integrator. On the other hand, if $1/s$ is the transmittance of only one of several incoming branches incident at the node V_k [as in Figure 4.14(b)], then V_k is *not* the output of an integrator. In order to identify the integrator outputs clearly for the purpose of circuit interconnection, we insert some dummy branches with weight 1 in series with the branches weighted $1/s$. When this is done to Figure 4.14(b), the result is Figure 4.15 with the inserted dummy branches shown in heavy lines. An SFG branch with weight $-1/s$ represents an inverting integrator. As pointed out in Section 4.2, the circuitry of an inverting integrator is simpler than that of a noninverting integrator. To have an implementation utilizing inverting integrators, we replace all SFG branch weights $1/s$ in Figure 4.14 by $-1/s$. In order to

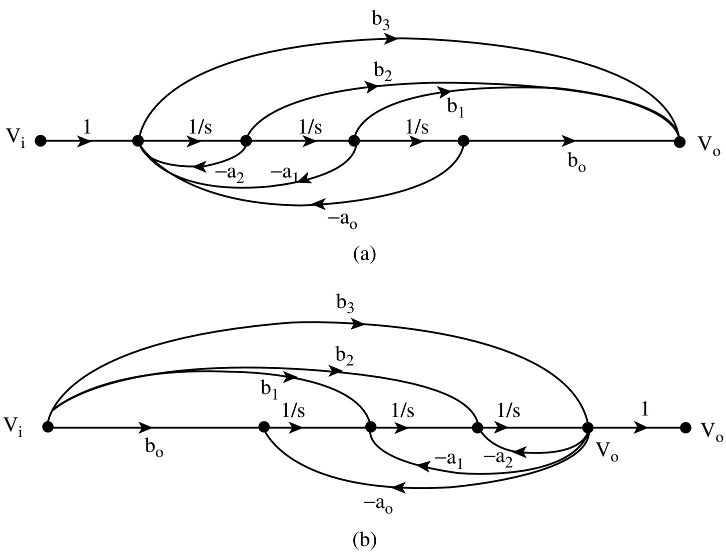


FIGURE 4.14 Two simple SFGs having a gain function given by Eq. (4.3). (a) Controllable canonical form. (b) Observable canonical form.

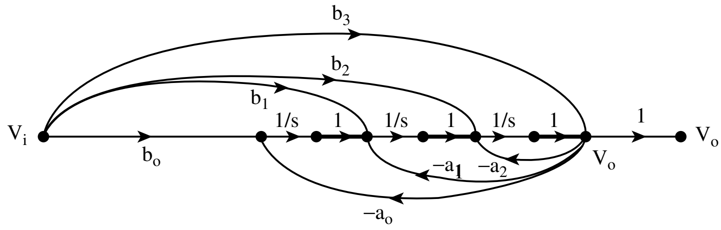


FIGURE 4.15 Insertion of dummy branches to identify integrator outputs.

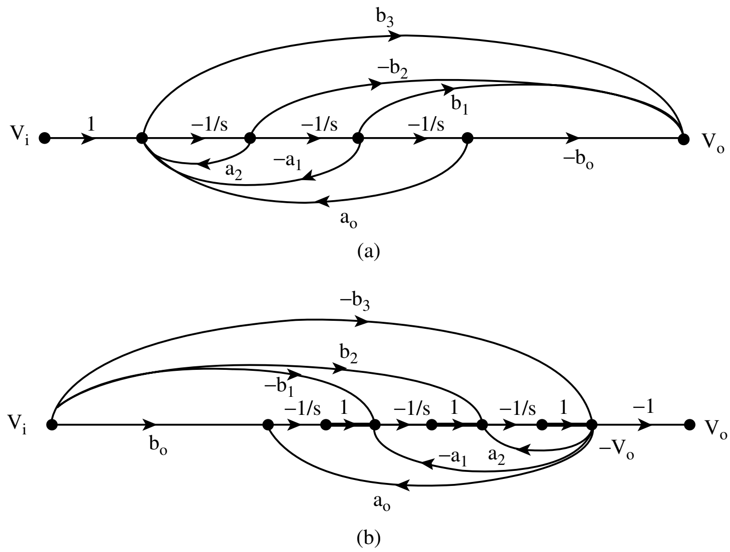


FIGURE 4.16 Simulation of $H(s)$ by an SFG containing inverting integrators.

maintain the original path and loop weights, the signs of some feedback branches and forward path branches must be changed accordingly. When this is done, Figure 4.14(a) and Figure 4.15 become those shown in Figure 4.16(a) and (b), respectively. Our next goal is to implement these SFGs by RC-op-amp circuits. Because SFGs of the kind described in this section are widely used in the study of linear systems by the state variable approach, the active filters based such SFGs are called *state variable filters* [6].

Example 5. Synthesize a state variable active filter to have a third order Butterworth low-pass response having 3 dB frequency $\omega_o = 10^6$ rad/s. All op amps used are single-ended.

Solution. As usual in filter synthesis, we first construct the filter for the normalized case, i.e., $\omega_o = 1$ rad/s, and then perform frequency scaling to obtain the required filter. The normalized voltage gain function of the filter is

$$H(s) = \frac{V_o}{V_i} = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{4.6}$$

and the two SFGs in Figure 4.16 become those depicted in Figure 4.17.

Because we are concerned with the magnitude response only, $-V_o$ instead of V_o can be accepted as the desired output. Therefore, in Figure 4.17, the rightmost SFG branch with gain (−1) need not be implemented. The implementation of the SFG as RC-op-amp circuits is now just a matter of looking up Table 4.2, selecting proper component networks and connecting them as specified by Figure 4.17. The

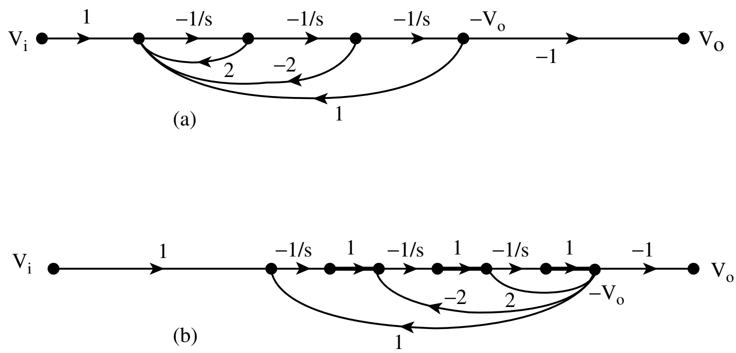
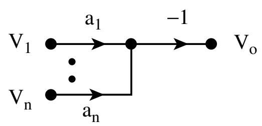
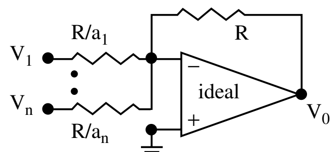
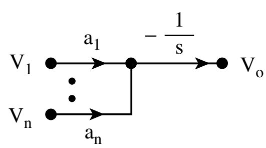
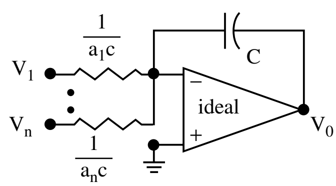
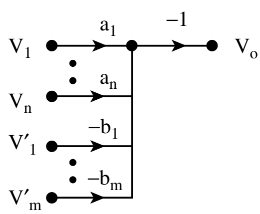
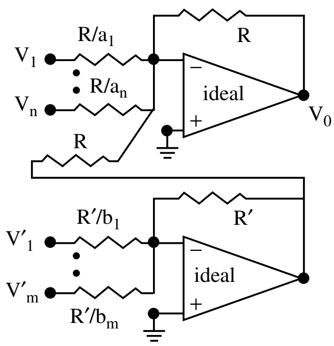
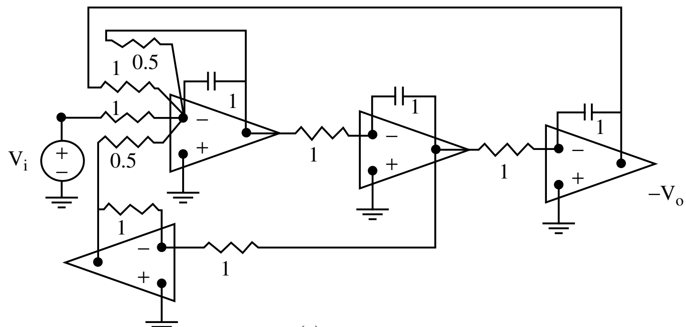


FIGURE 4.17 Two SFG representations of Eq. (4.6).

TABLE 4.2 Single-Ended Op Amp Circuits for Implementing State Variable Active Filters

Signal flow graph	RC-op-amp circuit
(1)  $V_o = -(a_1V_1 + \dots + a_nV_n)$	Inverting scaled summer  R: arbitrary
(2)  $V_o = -\frac{1}{s}(a_1V_1 + \dots + a_nV_n)$	Inverting scaled summing integrator  C: arbitrary
(3)  $V_o = -(a_1V_1 + \dots + a_nV_n) + (b_1V'_1 + \dots + b_mV_n)$	Bi-polarity summer  R and R′: arbitrary



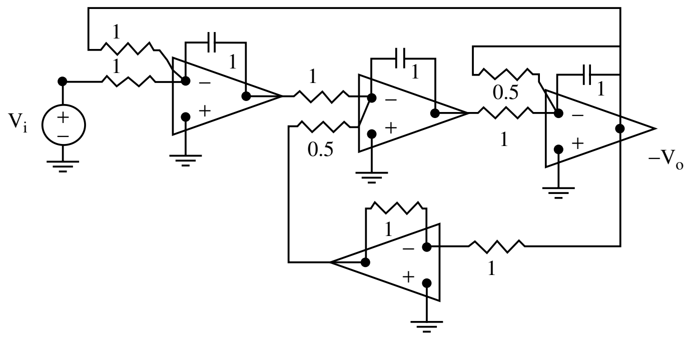


FIGURE 4.18 Two op amp circuit realizations of $H(s)$ given by Eq. (4.6).

results are given in Figure 4.18(a) and (b). These circuits, with element values in ohms and farads, realize the normalized transfer function having $\omega_c = 1$ rad/s. To meet the original specification of $\omega_c = 10^6$ rad/s, we frequency-scale the circuits by a factor 10^6 (i.e., divide all capacitances by 10^6). To have practical resistance values, we further magnitude-scale the circuits by a factor of, say, 1000. The resistances are multiplied by 1000, and the capacitances are further divided by 1000. The final circuits are still given by Figure 4.18, but now with element values in kΩ and nF.

In example 5, both realizations require 4 op amps. In general, for an nth order transfer function given by Eq. (4.1) with all coefficients nonzero, a synthesis based on Figure 4.16(a) (controllable canonical form) requires $n + 3$ single-ended op amps. The breakdown is as follows [refer to Figure 4.16(a)]:

n inverting scaled integrators (item 2, Table 4.2) for the n SFG branches with weight $-1/s$
2 op amps for the bipolarity summer (item 3, Table 4.2) to obtain V_o
1 inverting scaled summer (item 1, Table 4.2) to invert and add up signals from branches with weights $-a_1$, $-a_3$, etc., before applying to the left-most integrator

On the other hand, a synthesis based on Figure 4.16(b) (observable canonical form) requires only $n + 2$ single-ended op amps. To see this, we redraw Figure 4.16(b) as Figure 4.19 by inserting branches with weight −1, and making all literal coefficients positive. The breakdown is as follows (referring to Figure 4.19, extended to nth order $H(s)$):

n inverting scaled integrators (item 2, Table 4.2) for the n SFG branches with weight $-1/s$
1 inverting amplifier at the input end to provide $-V_i$
1 inverting amplifier at the output end to make available both V_o and $-V_o$

The number of op amps can be reduced if the restriction of using single-ended op amp is removed. Table 4.3 describes several differential-input op amp circuits suitable for use in the state variable active filters.

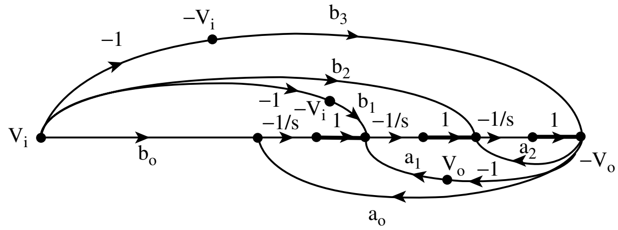
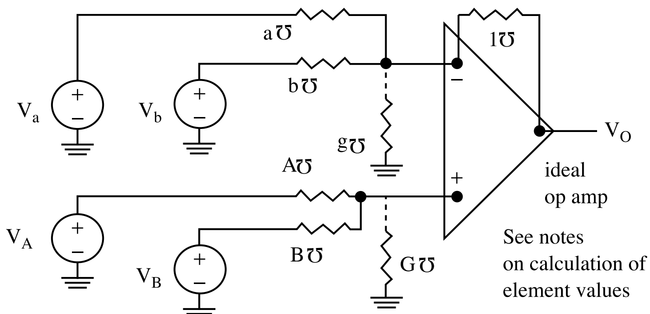


FIGURE 4.19 A modification of Figure 4.16(b) to use all positive *a*'s and *b*'s.

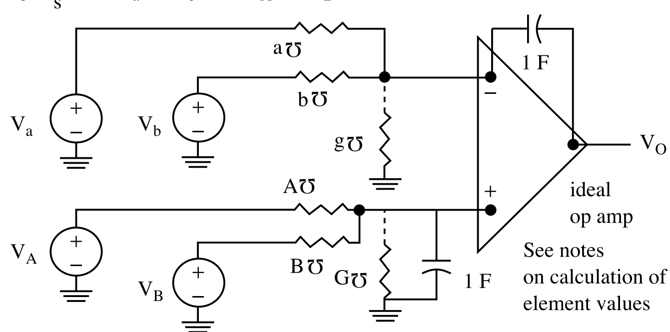
TABLE 4.3 Differential-Input Op Amp Circuit

(1) Bi-polarity scaled summer $V_o = -(aV_a + bV_b) + (AV_A + BV_B)$



(2) Bi-polarity scaled summing integrator

$$V_o = \frac{1}{s}\left[-(aV_a + bV_b) + (AV_A + BV_B)\right]$$



Note: Calculation of element values in Table 4.3[7].

(i) The initial design uses 1Ω resistance or 1 F capacitance as the feedback element.

(ii) Either the g mho conductance or the G mho conductance (not both) is connected. Choose the values of g or G such that the sum of all conductances connected to the inverting input terminal equals the sum of all conductances connected to the noninverting input terminal.

(iii) Starting with the initial design, one may magnitude-scale all elements connected to the inverting input terminal by one factor, and all elements connected to the noninverting input terminal by the same or a different factor.

If differential-input op amps are used, then the number of op amps required for the realization of Eq. (4.1) (with $m = n$) is reduced to $(n + 1)$ for Figure 4.16(a) and n for Figure 4.16(b). The breakdowns are as follows:

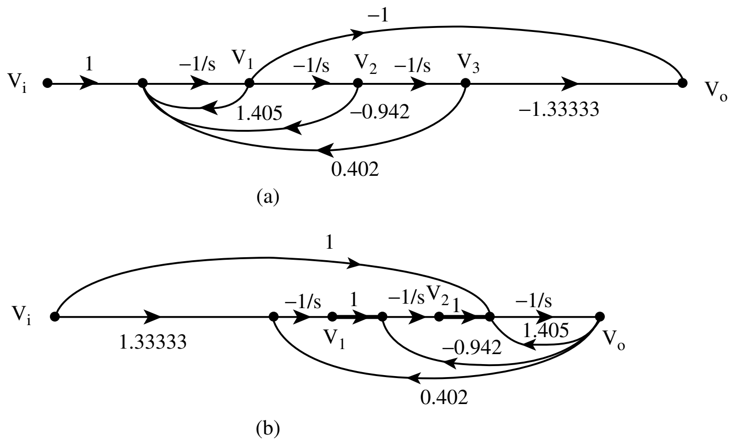


FIGURE 4.20 Two SFGs realizing the transfer function of Eq. (4.7).

For the controllable canonical form SFG of Figure 4.16(a):

$n - 1$ inverting integrators (item 2, Table 4.2 with one input) for the n SFG branches with weight $-1/s$, except the leftmost

1 bipolarity-scaled summing integrator (item 2, Table 4.3) for the leftmost SFG branch with weight $-1/s$

1 bipolarity-scaled summer (item 1, Table 4.3) to obtain V_o

For the observable canonical form SFG of Figure 4.16(b):

n bipolarity scaled summing integrator (item 2, Table 4.3), one for each SFG branch with weight $-1/s$

To construct the op amp circuit, one refers to the SFG of Figure 4.14 and obtains the expression relating the output of each op amp to the outputs of other op amps. After that is done, refer to Table 4.3, pick the appropriate component circuits, and connect them as specified by the SFG. The next example outlines the procedure of utilizing differential-input type op amps to reduce the total number of op amps to $(n + 1)$ or n.

Example 6. Design a state-variable active low-pass filter to meet the following requirements: magnitude response is of the inverse Chebyshev type

$$\alpha_{max} = 0.5 \text{ dB}, \qquad \alpha_{min} = 20 \text{ dB}, \qquad \alpha(\omega_s) = \alpha_{min}$$

$$\omega_p = 333.33 \text{ rad/s}, \qquad \omega_s = 1000 \text{ rad/s}$$

Solution. Using the method described in Chapter 71, the *normalized* transfer function (i.e., $\omega_s = 1$ rad/s) is found to be

$$H(s) = \frac{V_o}{V_i} = \frac{K(s^2 + 1.33333)}{s^3 + 1.40534s^2 + 0.94200s + 0.40196} \tag{4.7}$$

The SFGs for this $H(s)$ are simply obtained from Figure 4.16 by removing the two branches having weights b_3 and b_1.The results are shown in Figure 4.20(a) for the case $K = 1$, and in Figure 4.20(b) for the case $K = -1$. A four-op-amp circuit for the normalized $H(s)$ may be constructed in accordance with the SFG of Figure 4.20(a). The component op amp circuits are selected from Table 4.2 and 4.3 in the following manner:

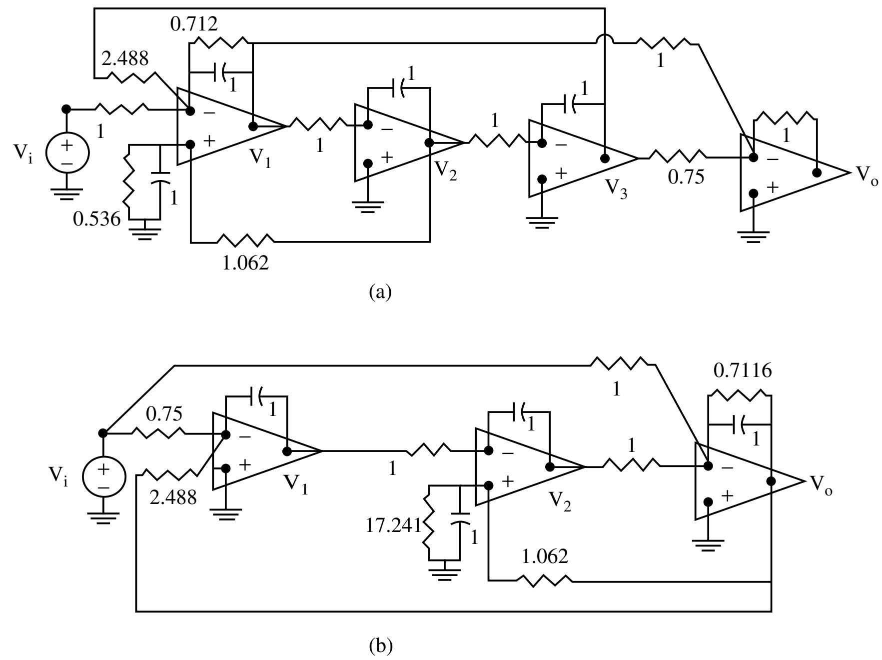


FIGURE 4.21 Two realizations of the third-order, inverse Chebyshev low-pass filter of Example 6. Element values are in kΩ and μF.

Relationship from SFG	Component op amp circuit
$V_1 = -\frac{1}{s}\left(V_i + 1.405V_1 + 0.402V_3 - 0.942V_2\right)$	item 2, Table 4.3
$V_2 = -\frac{1}{s}V_1$	item 2, Table 4.2
$V_3 = -\frac{1}{s}V_2$	item 2, Table 4.2
$V_o = -V_1 - 1.33333\ V_3$	item 1, Table 4.2

After connecting these four-component op amp circuits as described in Figure 4.20(a), and frequency-scaling the whole circuit by 1000, and magnitude-scaling by 1000, the final op amp circuit meeting the low-pass filter specifications is shown in Figure 4.21(a).

In a similar manner, a three-op-amp circuit for the normalized $H(s)$ may be constructed in accordance with the SFG of Figure 4.20(b). The final op amp circuit meeting the lowpass filter specifications is shown in Figure 4.21(b). Both circuits in Figure 4.21 achieve a gain constant $|K| = 1$ in Eq. (4.7). Should a different value of $|K| = 1/\alpha$ be desired, it is only necessary to multiply the values of all resistors connected to the input V_i by α.

When the method of this subsection is applied to a second order transfer function, the resultant op amp circuit is called a *state variable biquad*. Biquads and first order op amp circuits are used as the basic building blocks in the synthesis of a general nth order transfer function by the "cascade" approach. Depending on the SFGs chosen and the types of op amps allowed (single-ended or differential-input), a state variable biquad may require from 2 to 5 op amps. Some special but useful state variable biquads are listed in Table 4.4 for reference purposes.

TABLE 4.4 Some Special State-Variable Biquads

Normalized transfer functions

(1) Lowpass

$$\frac{V_{LP}}{V_i} = \frac{1}{s^2 + \frac{1}{Q}s + 1}$$

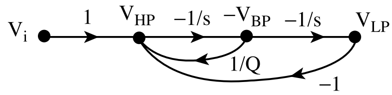
(2) Bandpass

$$\frac{V_{BP}}{V_i} = \frac{s}{s^2 + \frac{1}{Q}s + 1}$$

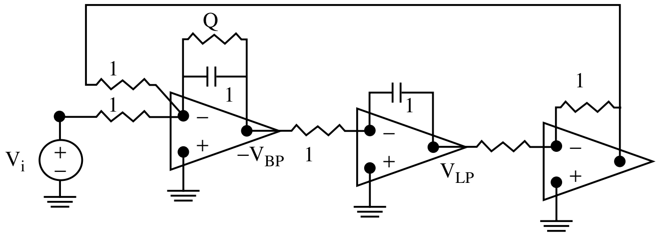
(3) Highpass

$$\frac{V_{HP}}{V_i} = \frac{s^2}{s^2 + \frac{1}{Q}s + 1}$$

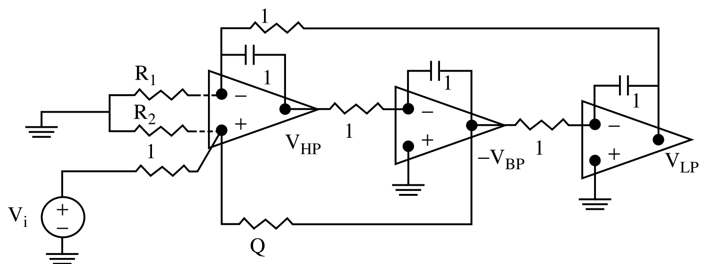
Signal flow graph for transfer functions (1) - (3)



Op amp circuits for (1) – (2). Available outputs: LP and BP



Op amp circuits for (1) – (3). Available outputs: LP, BP and HP



Either R1 or R2 is connected. See notes in Table 20.3 for the determination of their values.

All the SFGs used in the previous examples are of the two types (controllable and observable canonical forms) illustrated in Figure 4.16; however, many other possible SFGs produce the same transfer function. For example, a third-order, low-pass Butterworth or Chebyshev filter has an all-pole transfer function.

$$H(s) = \frac{V_o}{V_i} = \frac{K}{s^3 + a_2 s^2 + a_1 s + a_o} \tag{4.8}$$

A total of six SFGs may be constructed in accordance with Eq. (4.2) to produce the desired $H(s)$. These are illustrated in Figure 4.22. Among these, six SFGs only two have been chosen for consideration in this section.

Similarly, for a fourth-order, low-pass Butterworth or Chebyshev filter, a total of 20 SFGs may be constructed. The reader should consult References [8–9] for details.

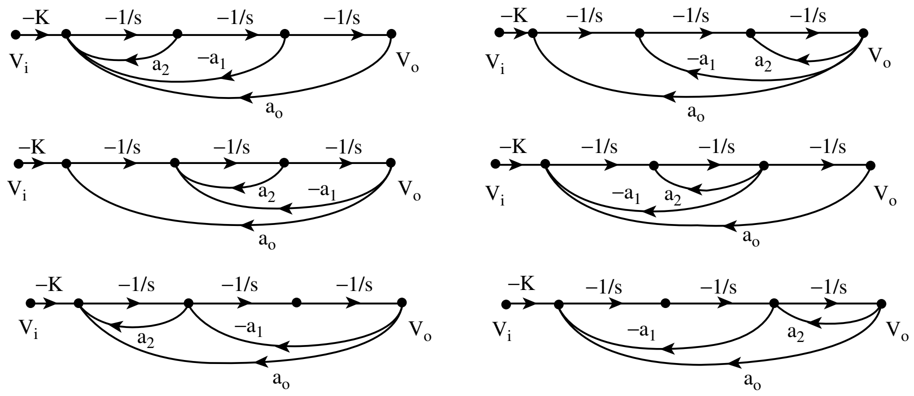


FIGURE 4.22 Six SFGs realizing a third-order, all-pole transfer function.

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