5

Analysis in the Frequency Domain

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5.1 Network Functions

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Definition of Network Functions

Network functions can be defined if the following constraints are satisfied:

1. The network is linear.
2. It is analyzed in the frequency domain using the Laplace transform.
3. All initial voltages and currents are zero (zero state conditions).

This chapter demonstrates how the various functions can be derived, but first we introduce some explanations and definitions. If we analyze any linear network, we can take as output any nodal voltage, or a difference of any two nodal voltages; denote such as output voltage by V_{out}. We can also take as the output a current through any element of the network; we call it output current, I_{out}. If the network is excited by a voltage source, E, then we can also calculate the current delivered into the network by this source; this is the input current, I_{in}. If the network is excited by a current source, J, then the voltage across the current source is the input voltage, V_{in}.

Suppose that we analyze the network and keep the letter E or J in our derivations. Then, we can define the following network functions:

$$\begin{aligned} &\text{Voltage transfer function,} && T_v = \frac{V_{out}}{E} \\ &\text{Input admittance,} && Y_{in} = \frac{I_{in}}{E} \\ &\text{Transfer admittance,} && Y_{tr} = \frac{I_{out}}{E} \end{aligned} \tag{5.1}$$

Current transfer function, $T_i = \dfrac{I_{out}}{J}$

Input impedance, $Z_{in} = \dfrac{V_{in}}{J}$

Transfer impedance, $Z_{tr} = \dfrac{V_{out}}{J}$

Output impedance or output admittance are also used, but the concept is equivalent to the input impedance or admittance. The only difference is that, for calculations, the source is placed temporarily at a point from which the output normally will be taken. In the Laplace transform, it is common to use capital letters, V for voltages and I for currents. We also deal with impedances, Z, and admittances, Y. Their relationships are

$$V = ZI \qquad I = YV$$

The impedance of a capacitor is $Z_C = 1/sC$, the impedance of an inductor is $Z_L = sL$, and the impedance of a resistor is R. The inverse of these values are admittances: $Y_C = sC$, $Y_L = 1/sL$, and the admittance of a resistor is $G = 1/R$.

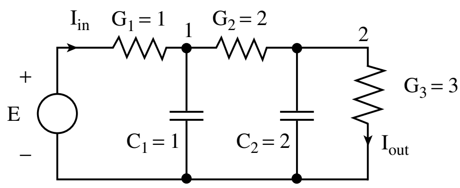

FIGURE 5.1

To demonstrate the derivations of the above functions two examples are used. Consider the network in Figure 5.1, with input delivered by the voltage source, E. By Kirchhoff's current law (KCL), the sum of currents flowing *away* from node 1 must be zero:

$$\left(G_1 + sC_1 + G_2\right)V_1 - G_2V_2 - EG_1 = 0$$

Similarly, the sum of currents flowing away from node 2 is

$$-V_1G_2 + \left(G_2 + sC_2 + G_3\right)V_2 = 0$$

The independent source is denoted by the letter E, and is assumed to be known. In mathematics, we transfer known quantities to the right. Doing so and collecting the equations into one matrix equation results in

$$\begin{bmatrix} G_1 + G_2 + sC_1 & -G_2 \\ -G_2 & G_2 + G_3 + sC_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} EG_1 \\ 0 \end{bmatrix}$$

If numerical values from the figure are used, this system simplifies to

$$\begin{bmatrix} s+3 & -2 \\ -2 & 2s+5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

or

$$\boldsymbol{YV} = \boldsymbol{E}$$

Any method can be used to solve this system, but for the sake of explanation it is advantageous to use Cramer's rule. First, find the determinant of the matrix,

$$D = 2s^2 + 11s + 11$$

To obtain the solution for the variable V_1 (V_2), replace the first (second) column of $\boldsymbol{Y}$ by the right-hand side and calculate the determinant of such a modified matrix. Denoting such a determinant by the letter N with an appropriate subscript, evaluate

$$N_1 = \begin{vmatrix} E & -2 \\ 0 & 2s+5 \end{vmatrix} = (2s+5)E$$

Then,

$$V_1 = \frac{N_1}{D} = \frac{2s+5}{2s^2+11s+11} E$$

Now, divide the equation by E, which results in the voltage transfer function

$$T_v = \frac{V_1}{E} = \frac{2s+5}{2s^2+11s+11}$$

To find the nodal voltage V_2, replace the second column by the elements of the vector on the right-hand side:

$$N_2 = \begin{vmatrix} s+3 & E \\ -2 & 0 \end{vmatrix} = 2E$$

The voltage is

$$V_2 = \frac{N_2}{D} = \frac{2}{2s^2+11s+11} E$$

and another voltage transfer function of the same network is

$$T_v = \frac{V_2}{E} = \frac{2}{2s^2+11s+11}$$

Note that many network functions can be defined for any network. For instance, we may wish to calculate the currents I_{in} or I_{out}, marked in Figure 5.1. Because the voltages are already known, they are used: For the output current $I_{out} = G_3 V_2$ and divided by E

$$Y_{tr} = \frac{I_{out}}{E} = \frac{3V_2}{E} = \frac{6}{2s^2+11s+11}$$

The input current $I_{in} = E - G_1 V_1 = E - V_1 = E(2s^2 + 9s + 6)/(2s^2 + 11s + 11)$ and dividing by E

$$Y_{in} = \frac{I_{in}}{E} = \frac{2s^2+9s+6}{2s^2+11s+11}$$

In order to define the other possible network functions, we must use a current source, J, as in Figure 5.2, where we also take the current through the inductor as an output variable. This method of formulating the network equations is called **modified nodal**. The sum of currents flowing away from node 1 is

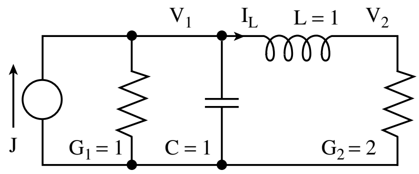

FIGURE 5.2

$$(G_1 + sC_1)V_1 + I_L - J = 0$$

from node 2 it is

$$G_2V_2 - I_L = 0$$

and the properties of the inductor are expressed by the additional equation

$$V_1 - V_2 - sLI_L = 0$$

Inserting numerical values and collecting in matrix form:

$$\begin{bmatrix} s+1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & -s \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_L \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the system is

$$D = -(2s^2 + 3s + 3)$$

To solve for V_1, we replace the first column by the right-hand side and evaluate the determinant

$$N_1 = \begin{vmatrix} J & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & -s \end{vmatrix} = -(2s+1)J$$

Then, $V_1 = N_1/D$ and dividing by J we obtain the network function

$$Z_{\text{tr}} = \frac{V_1}{J} = \frac{2s+1}{2s^2 + 3s + 3}$$

To obtain the inductor current, evaluate the determinant of a matrix in which the third column is replaced by the right-hand side: $N_3 = -2J$. Then, $I_L = N_3/D$ and

$$T_i = \frac{I_L}{J} = \frac{2}{s^2 + 3s + 3}$$

In general,

$$F = \frac{\text{Output variable}}{E \text{ or } J} = \frac{\text{Numerator polynomial}}{\text{Denominator polynomial}} \qquad (5.2)$$

Any method that may be used to formulate the equations will lead to the same result. One example shows this is true. Reconsider the network in Figure 5.2, but use the admittance of the inductor, $Y_L = 1/sL$, and do not consider the current through the inductor. In such a case, the nodal equations are

$$\left\{1+s+\frac{1}{s}\right\}V_1-\frac{1}{s}V_2=J$$

$$-\frac{1}{s}V_1+\left\{\frac{1}{s}+2\right\}V_2=0$$

We can proceed in two ways:

1. We can multiply each equation by s and thus remove the fractions. This provides the system equation

$$\begin{bmatrix} s^2+s+1 & -1 \\ -1 & 2s+1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=\begin{bmatrix} sJ \\ 0 \end{bmatrix}$$

   The determinant of this matrix is $D = 2s^3 + 3s^2 + 3s$. To calculate V_1, find $N_1 = s(2s + 1)J$. Their ratio is the same as before because one s in the numerator can be canceled against the denominator.
2. If we do not remove the fractions and go ahead with the solution, we have the matrix equation

$$\begin{bmatrix} s+1+1/s & -1/s \\ -1/s & 1/s+2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=\begin{bmatrix} J \\ 0 \end{bmatrix}$$

   The determinant is $D = 2s + 3 + 3/s$ and the numerator for V_1 is $N_1 = (1/s + 2)J$. Taking their ratio

$$V_1=\frac{(1/s+2)J}{2s+3+3/s}=\frac{2s+1}{2s^2+3s+3}J$$

   which is the same result as before.

We conclude that it does not matter which method is used to formulate the equations. The result is always a ratio of two polynomials in the variable s.

Many additional conclusions can be drawn from these examples. The most important result so far is that **all network functions of any given network have the same denominator**. It was easy to discover this property because we used Cramer's rule, with its evaluation by the ratio of two determinants. It should be mentioned at this point that we may have network functions in which some terms of the numerator can cancel against the same terms of the denominator. Such a cancellation represents a mathematical simplification which does not change the validity of the above statement.

Occasionally, the network may have more than one source. In such cases, we apply the superposition principle of linear networks. The contribution to the output can be calculated separately for each source and the results added. All that must be done is to correctly remove those sources which are not considered at the moment. All *unused independent voltage sources* must be replaced by *short circuits*. All *unused independent current sources* are replaced by *open circuits* (removed from the network). Although we did not use dependent sources in our examples, it is necessary to stress that such removal *must not* take place for dependent sources.

Network functions can be used to find responses to any given input signal. First, multiply the network function by E or J; this will give the expression for the output. Afterward, the letter E or J is replaced by the Laplace transform of the signal. For instance, if the signal is a unit step, then the source is replaced by $1/s$. If it is cost ωt, then the source is replaced by the Laplace transform, $s/(s^2 + \omega^2)$, and so on.

In the Laplace transform, one special signal exists, the Dirac impulse, commonly denoted by $\delta(t)$. It can be represented as a rectangular pulse having width T and height $1/T$. The area of the pulse is always 1, even if we go to $\lim T \to 0$, which is the Dirac impulse. Its Laplace transform is 1. Because multiplication by 1 does not change the network function, we conclude that any network function is also the Laplace transform of the network response to the Dirac impulse.

A word of caution: In the network function always divide by the *independent* voltage (current) source. We cannot take two analysis results, for instance V_1 and V_2, derived for Figure 5.1, and take their ratio. This will not be a network function.

Poles and Zeros

Networks with lumped elements have network functions which are always ratios of two polynomials with real coefficients. For some applications the polynomials may be expressed as functions of some (or all) elements, but the principle is unchanged.

Because we have a ratio of two polynomials, the network function can be written in two forms:

$$F = \frac{\sum_{i=o}^{M} a_i s^i}{\sum_{i=o}^{N} b_i s^i} = K \frac{\prod_{i=1}^{M} (s - z_i)}{\prod_{i=1}^{N} (s - p_i)} \tag{5.3}$$

The middle form is what we obtain from analyses similar to those in the examples. Algebraically, a polynomial of order N has exactly N roots. This leads to the form on the right. The multiplicative constant, K, is the ratio

$$K = \frac{a_M}{b_N}$$

and is obtained by dividing each polynomial by the coefficient of its highest power.

It is easy to find roots of a first- and second-order polynomial because formulas are available, but in all other cases iterative methods and a computer are utilized. However, even without actually finding the roots, we can draw a number of important conclusions.

If the highest power of the polynomial is odd, then at least one real root will exist. The other roots may be either real or complex, but if they are complex, then they always appear in complex conjugate pairs. The roots of the numerator are called *zeros*, and those of the denominator are called *poles*. We denote the zeros by

$$z_i = a_i + jb_i$$

where $j = \sqrt{-1}$. Either a or b may be zero. For the poles, we have similarly

$$p_i = c_i + jd_i$$

The polynomial also may have multiple roots. For instance, the polynomial $P(s) = (s + 1)^2 (s + 2)^3$ has a double root at $s = -1$ and a triple root at $s = -2$. The positions of the poles and zeros, with the constant K, completely define the network function and also all network properties. The positions can be plotted in a complex plane, the zeros indicated by small circles and poles by crosses. A multiple pole (zero) is indicated by a number appearing at the cross (circle). Figure 5.3 shows a network function with two complex conjugate zeros on the imaginary axis, two complex conjugate poles, and one double real pole.

As derived previously, all network functions of any given network have the same poles. Their positions depend only on the structure of the network and are independent of the signal or where the signal is applied. Because of this fundamental property, the poles are also called *natural frequencies* of the network.

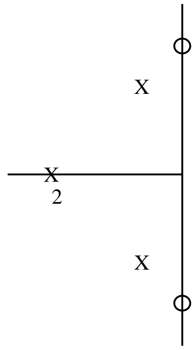

FIGURE 5.3

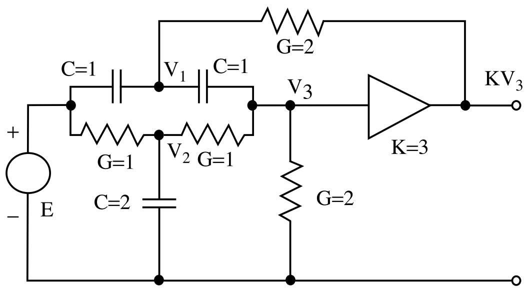

FIGURE 5.4

The zeros depend on the place at which we attach the source and also on the point where we take the output.

It is possible to have networks in which a pole is in exactly the same position as a zero; mathematically, such terms cancel. Figure 5.4 is an example. Writing the sum of currents at nodes 1, 2, and 3, we obtain

$$(2s+2)V_1-(s+3)V_3=sE$$

$$(2s+2)V_2-V_3=E$$

$$-sV_1-V_2+(s+3)V_3=0$$

and in matrix form

$$\begin{bmatrix} 2s+2 & 0 & -(s+3) \\ 0 & 2s+2 & -1 \\ -s & -1 & s+3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} sE \\ E \\ 0 \end{bmatrix}$$

By carefully evaluating the determinant we discover that we can keep the term $(2s + 2)$ separate and get $D = (2s + 2)(s^2 + 2s + 5)$. Replacing the third column by the right-hand side, we calculate the numerator $N_3 = (2s + 2)(s^2 + 1)E$. Because the output is KV_3, the voltage transfer function is

$$T_v=\frac{3(2s+2)(s^2+1)}{(2s+2)(s^2+2s+5)}$$

Mathematically, the term $(2s + 2)$ cancels and the network function is sometimes written as

$$T_v = \frac{3(s^2+1)}{s^2+2s+5}$$

Such cancellation makes the denominator different from other network functions that we might derive for the same network, but it is not a correct way to describe the properties of the network. The cancellation gives the impression that we have a second-order network, while it is actually a third-order network.

Network Stability

Stability of the network depends entirely on the positions of its poles. The following is a list of the conditions in order for the network to be stable, with subsequent explanation of the reasons:

1. The network is stable if all its poles are in the left half of the complex plane.
2. The network is unstable if at least one of its poles is in the right-half plane.
3. The network is marginally stable if all its poles are simple and exactly on the imaginary axis.
4. The network is unstable if it has all poles on the imaginary axis, but at least one of them has multiplicity two or more.

Courses on mathematics teach the process of decomposing a rational function into partial fractions. We show an example with one simple real pole and a pair of simple complex conjugate poles,

$$F(s) = \frac{3s^2+8s+6}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} + \frac{1+j}{s+1+j} + \frac{1-j}{s+1-j}$$

The poles are $p_1 = -1$ and $p_{2,3} = -1 \pm j$, all with negative real parts and all lying in the left-half plane. Partial fraction decomposition is on the right of the preceding equation. It is always true, for any lumped network, that the decomposition for a real pole has a real constant in the numerator. Complex poles always appear in complex conjugate pairs and the decomposition constants, if complex, also are complex conjugate. Once such a decomposition is available, tables can be used to invert the functions into time domain. The decomposition may be quite a laborious process, however, only a few types of terms need be considered for lumped networks. All are collected in Table 5.1. Each time domain expression is multiplied by unit step, $u(t)$, which is zero for $t < 0$ and is one for $t \geq 0$. Such multiplication correctly expresses the fact that the time functions start at $t = 0$.

Formula one in Table 5.1 shows that a real, single pole in the left-half plane will lead to a time-domain function which decreases as e^{-ct}. This response is called stable. If $c = 0$, then the response becomes $u(t)$.

TABLE 5.1

Formula	Laplace Domain	Time Domain
1	$\frac{K}{s+c}$	$Ke^{-ct}u(t)$
2	$\frac{K}{(s+c)^n}$	$K\frac{t^{n-1}}{(n-1)!}e^{-ct}u(t)$
3	$\frac{A+jB}{s+c+jd}+\frac{A-jB}{s+c-jd}$	$2e^{-ct}(A\cos dt + B\sin dt)u(t)$
4	$\frac{A+jB}{(s+c+jd)^n}+\frac{A-jB}{(s+c-jd)^n}$	$\frac{2t^{n-1}}{(n-1)!}e^{-ct}(A\cos dt + B\sin dt)u(t)$

Should the pole be in the right-half plane, the exponent will be positive and e^{ct} will grow rapidly and without bound. This network is said to be unstable.

Formula two shows what happens if the pole is real, with multiplicity n. If it is in the left-half plane, then t^{n-1} is a growing function, but e^{-ct} decreases faster, and for large t the result tends to zero. The function is still stable.

Formula three considers the case of two simple complex conjugate poles. Their real parts influence the exponent, and the imaginary parts contribute to oscillations. If the real part is negative, the oscillations will be damped, the response will become zero for large t, and the network will be stable. If the real part is zero, then the oscillations continue indefinitely with constant amplitude. For the positive real part, the network becomes unstable.

Formula four considers a pair of multiple complex conjugate poles. As long as the real part is negative, the oscillations will decrease with time and the network will be stable. If a real part is zero or positive, the network is unstable because the oscillations will grow.

Initial and Final Value Theorems

Finding the poles and evaluating the time domain response is a complicated process, which normally requires the use of a computer. It is, therefore, advisable to use all possible steps that may provide information about the network behavior without actually finding the full time-domain response.

Two Laplace transform theorems help in finding how the network behaves at $t = 0$ and at $t \to \infty$. Both theorems are derived from the Laplace transform formula for differentiation,

$$\int_{0^-}^{\infty} f'(t)e^{-st}dt = sF(s) - f(0^-) \tag{5.4}$$

where 0^- indicates that we are considering the instant just before the signal is applied. If we let $s \to 0$, then $e^0 = 1$, and the integral of the derivative becomes the function itself. Inserting the integrating limits we get

$$f(\infty) - f(0^-) = \lim_{s\to 0}\left[sF(s) - f(0^-)\right]$$

Cancelling $f(0^-)$ on both sides, we arrive at the *final value theorem*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{5.5}$$

Another possibility is to let $s \to \infty$; then e^{-st} in (5.4) will be zero and the whole left side becomes zero. This can be written as

$$0 = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

and because $f(0^-)$ is nothing but the limit of $f(t)$ for $t \to 0^-$, we obtain the *initial value theorem*

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \tag{5.6}$$

Note the similarity of the two theorems; we will apply them to the function used in the previous section. Consider

$$sF(s) = \frac{3s^3 + 8s^2 + 6s}{s^3 + 3s^2 + 4s + 2}$$

TABLE 5.2

Laplace Domain $D = s^3 + 3s^2 + 4s + 2$	Time Domain	$s \to \infty$ $t = 0$	$s = 0$ $t \to \infty$
$F(s) = 1/D$	$f(t) = e^{-t}(1 - \cos t)u(t)$	0	0
$G(s) = sF(s) = s/D$	$g(t) = f'(t) = e^{-t}(-1 + \cos t + \sin t)u(t)$	0	0
$H(s) = s^2F(s) = s^2/D$	$h(t) = f''(t) = e^{-t}(1 - 2\sin t)u(t)$	1	0
$K(s) = s^3/D$	$k(t) = f'''(t) = \delta(t) + e^{-t}(2\sin t - 2\cos t - 1)u(t)$	$\delta(t)$	0

If we take any large value of s, the highest powers will dominate and in limit, for $s \to \infty$, we get 3. This is the value of the time-domain response at $t = 0$. The limit for $s = 0$ is zero, and from the final value theorem we know that $f(t)$ will be zero for $t \to \infty$.

To extract still more information, use the example collected in Table 5.2. Scrolling down the table, each Laplace domain function is s times that above it. Each multiplication by s means differentiation in the time domain, as follows from (5.4). Scrolling down the second column of Table 5.2, each function is the derivative of that above it. To apply the limiting theorems, take the Laplace domain formula, which is one level lower, and insert the limits. The limiting is also shown and is confirmed by inserting either $t = 0$ or $t \to \infty$ into the time functions.

Although the two theorems are useful, the final value theorem is valid *only if the function is stable*. Consider the unstable function with two poles in the right-half plane

$$F_1(s) = \frac{1}{(s+1)(s-1+j)(s-1-j)} = \frac{1}{s^3 - s^2 + 2}$$

Its time-domain response is

$$f_1(t) = \frac{1}{5}\left[e^{-t} + e^{+t}\left(2\sin^t - \cos^t\right)\right]u(t)$$

and the term e^{+t} will cause the function to grow for large t. If the final value theorem is applied, we consider

$$sF_1(s) = \frac{s}{s^3 + s^2 + 2}$$

Inserting $s = 0$, the theorem predicts that the time function will approach zero for large t. This is disappointing, but some additional simple rules can be applied. The function is unstable if some coefficients of the denominator are missing, or if the denominator coefficients do not all have the same sign (all + or all –). Such situations are easily detected, but if all coefficients have the same sign, nothing can be said about stability. Additional theorems exist (e.g., Hurwitz theorem), but if in doubt, it is probably simplest to go to the computer and find the poles.

5.2 Advanced Network Analysis Concepts

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Introduction

The systematic analysis of an electrical or electronic network entails formulating and solving the relevant Kirchhoff equations of equilibrium. The analysis is conducted to acquire a theoretically sound understanding of circuit responses. Such an understanding minimally delineates the dynamical effects of topology, controllable circuit branch variables, and observable parameters for active devices embedded in the circuit. It also illuminates circuit node and branch impedances to which the relevant responses of the circuit undergoing investigation are especially sensitive. Unfortunately, the complexity of modern networks, and particularly integrated analog electronic circuits, often inhibits the mathematical tractability that underpins an engineering understanding of circuit behavior. It is therefore not surprising that when mathematical analyses accompany a computer-assisted circuit design venture, the subcircuits identified for manual study are simplified representations of the corresponding subcircuits in the draft design solution. Unless care is exercised, these approximations can mask a satisfying understanding, and they can even lead to erroneous results.

Analytical and modeling approximations notwithstanding, the key to assimilating a satisfying understanding of the electrical characteristics of complex circuits is appropriate studies of simpler partitions of these circuits. To this end, Kron [1, 2] has provided, and others have explained and reinforced [3–5], an elegant theory that allows the circuit response solutions of these network partitions to be coalesced so that the desired response of the interconnected circuit is reconstructed exactly. Aside from formalizing an analytical mechanism for studying complicated circuits in terms of the solutions gleaned for more manageable subcircuits of the composite network [6], Kron's work allows for a computationally efficient study of feedback network responses. The theory also allows for the investigation of the sensitivity of overall network performance with respect to both small and large parametric changes [7]. In view of the exclusive focus on linear circuits in this section, it is worth interjecting that a form of Kron's partitioning theory is also applicable to certain classes of nonlinear circuits [8].

Fundamental Network Analysis Concepts

The derivation of Kron's formula, as well as the development of a general methodology for applying Kron's partitioning mechanism to the analyses of complex circuits, requires a fundamental understanding of the classical techniques exploited in the analysis of linear networks. Such an understanding begins by considering the $(n + 1)$ node, b branch linear network abstracted in Figure 5.5(a). The input port, which is defined by the node pair, 1-2, is excited by a signal source whose Thévenin voltage is V_s and whose Thévenin impedance is Z_s. In response to this excitation, a load voltage, V_L, is developed across a load impedance, Z_L, which shunts the output port consisting of the node pair, 3-4. Two other nodes, nodes m and p, are explicitly delineated for future reference. In response to the applied signal source, the voltage across the input port is V_I, while the voltage established across the node pair, m-p is V_k. In addition, the reference, or ground, node is labeled node 0. Either node 2, node 4, or both of these nodes can be incident with the ground node; that is, the signal source and/or the load impedance can be terminated to the

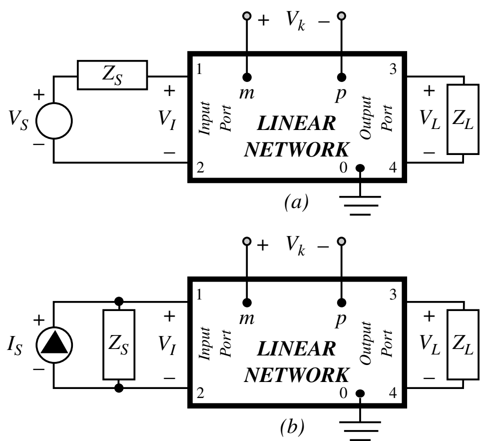

FIGURE 5.5 (a) Generalized linear network driven by a voltage source. (b) The network of (a), but with the signal excitation represented by its Norton equivalent circuit.

network ground. The diagram in Figure 5.5(b) is identical to that of Figure 5.5(a) except for the fact that the applied signal source is represented by its Norton equivalent circuit, where the Norton signal current, I_s, is

$$I_S = \frac{V_S}{Z_S} \tag{5.7}$$

Assuming that a nodal admittance matrix exists for the linear $(n + 1)$ node network at hand, the n equilibrium KCL equations can be expressed as the matrix relationship

$$\mathbf{J} = \mathbf{YE} \tag{5.8}$$

where $\mathbf{J}$ is an n-vector whose ith entry, J_i, is an independent current flowing into the ith circuit node, $\mathbf{E}$ is an n-vector of node voltages such that its ith entry, E_i, is ith node voltage referenced to network ground, and $\mathbf{Y}$, a square matrix of order n, is the nodal admittance matrix of the circuit. If $\mathbf{Y}$ is nonsingular, the node voltages follow as

$$\mathbf{E} = \mathbf{Y}^{-1}\mathbf{J} \tag{5.9}$$

Note that (5.9) is useful symbolically, but not necessarily computationally. In particular, (5.9) shows that the n node voltages of the $(n + 1)$ node, b branch network of Figure 5.5 can be straightforwardly computed in terms of the known independent current source vector and the parameters embedded in the network nodal admittance matrix. In an actual analytical environment, however, the nodal admittance matrix is rarely formulated and inverted. Instead, some or all of the n node voltages of interest are determined merely by algebraically manipulating and solving either the n independent KCL equations or the $(b - n)$ independent Kirchhoff's voltage law (KVL) equations that are required to establish the equilibrium conditions of the subject network.

If the n vector, $\mathbf{E}$, is indeed evaluated, all n independent node voltages are known, because

$$\mathbf{E}^{\mathrm{T}} = \left[E_1, E_2, E_3, \ldots, E_m, \ldots, E_p, \ldots, E_n\right] \tag{5.10}$$

where the superscript T indicates the operation of *matrix transposition*. In general, E_i, for $i = 1, 2, \ldots, n$, is the voltage developed at node i with respect to ground. It follows that the voltage between any two nodes derives directly from the network solution inferred by (5.9). For example, the input port voltage, V_I, is $(E_1 - E_2)$, the output port voltage, V_L, is $(E_3 - E_4)$, and the voltage, V_k, from node m to node p is $V_k = (E_m - E_p)$.

The calculation of the voltage appearing between any two circuit nodes can be formalized with the help of the generalized network diagrammed in Figure 5.6 and through the introduction of the *connection vector* concept. In particular, let $\mathbf{A}_{ij}$ denotes the $(n \times 1)$ connection vector for the port defined by the node pair, i-j. Moreover, let the voltage, V, at node i be taken as positive with respect to node j, and allow a current, I (which may be zero), to flow into node i and out of node j, as indicated in the diagram. Then, the elements of the connection vector, $\mathbf{A}_{ij}$, are all zero except for a + 1 in its ith row and a – 1 in

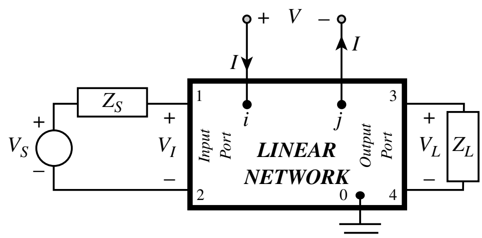

FIGURE 5.6 Generalized network diagram used to define the connection vector concept.

its jth row. If node j is the reference node, all elements of $\mathbf{A}_{ij}$, which in the case can be written simply as $\mathbf{A}_i$, are zero except for the ith row element, which remains +1. Thus, $\mathbf{A}_{ij}$ has the form

$$\mathbf{A}_{ij}^{\mathrm{T}} = \begin{array}{c} \\ \\ \end{array}\begin{matrix} & & & i\text{th column} & & & & n\text{th column} \\ & & & \downarrow & & & & \downarrow \\ [0 & 0 & \cdots & +1 & \cdots & -1 & \cdots & 0] \\ & & & & & \uparrow & & \\ & & & & & j\text{th column} & & \end{matrix} \tag{5.11}$$

For the special case in which a circuit branch element interconnects every pair of circuit nodes, $\mathbf{A}_{ij}$ is the appropriate column of the *node to branch incidence matrix*, which is a rectangular matrix of order $(n \times b)$, for the $(n + 1)$ node, b branch network at hand [9].

Returning to the calculation of V_I, V_L, and V_k, it follows from (5.9) through (5.11) that

$$V_I = E_1 - E_2 = \mathbf{A}_{12}^{\mathrm{T}}\mathbf{E} = \mathbf{A}_{12}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{J} \tag{5.12}$$

$$V_L = E_3 - E_4 = \mathbf{A}_{34}^{\mathrm{T}}\mathbf{E} = \mathbf{A}_{34}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{J} \tag{5.13}$$

and

$$V_k = E_m - E_p = \mathbf{A}_{mp}^{\mathrm{T}}\mathbf{E} = \mathbf{A}_{mp}^{\mathrm{T}}\mathbf{Y}^{1}\mathbf{J} \tag{5.14}$$

Assuming that I_s is the only independent source of excitation in the network of Figure 5.5

$$\mathbf{J} = \mathrm{A}_{12}I_S \tag{5.15}$$

which is the mathematical equivalent of the observation that the Norton source current, I_S, is entering node 1 of the network and leaving node 2. Accordingly,

$$V_I = \left(\mathbf{A}_{12}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right)I_S \tag{5.16}$$

$$V_L = \left(\mathbf{A}_{34}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right)I_S \tag{5.17}$$

and

$$V_k = \left(\mathbf{A}_{mp}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right)I_S \tag{5.18}$$

Several noteworthy features are implicit to the foregoing three relationships. First, each of the three parenthesized matrix products on the right-hand sides of the equations is a scalar. This observation follows from the facts that a transposed connection vector is a row matrix of order $(1 \times n)$, the inverse nodal admittance matrix is an n-square, and a connection vector is an n-vector. Second, these scalar products represent transimpedances from the input port to the port at which the voltage of interest is extracted. In the case of (5.16), the ratio, $V_{\mathrm{I}}\, I_S$, is actually the impedance, Z_{SS}, seen by the Norton current, I_S; that is,

$$Z_{SS} \stackrel{\Delta}{=} \frac{V_S}{I_S} = \left(\mathbf{A}_{12}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right) \tag{5.19}$$

where, as asserted previously, I_S is presumed to be the only source of energy applied to the network undergoing study. Similarly,

$$Z_{LS} \stackrel{\Delta}{=} \frac{V_L}{I_S} = \left(\mathbf{A}_{34}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right) \tag{5.20}$$

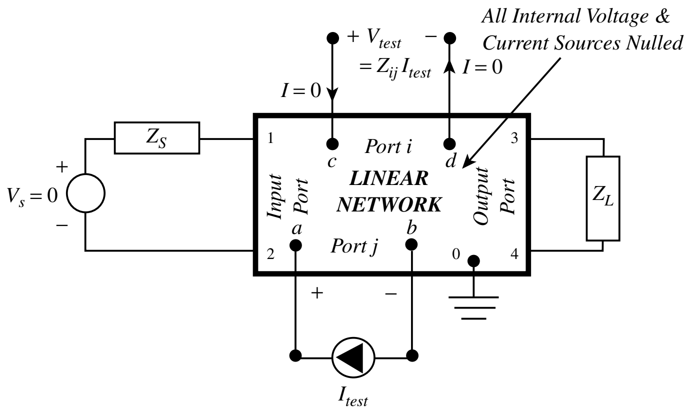

FIGURE 5.7 An illustration of a practical manual technique for computing the transimpedance between any port *j* to any port *i* of a linear network.

is the transimpedance from the input port to the output port, while

$$Z_{kS} \stackrel{\Delta}{=} \frac{V_k}{I_S} = \left(\mathbf{A}_{mp}^{\mathrm{T}} \mathbf{Y}^{-1} \mathbf{A}_{12}\right) \tag{5.21}$$

is the transimpedance from the input port to the port defined by the node pair, *m-p*.

The impedance in (5.19) and the transimpedances given by (5.20) and (5.21) are cast as explicit algebraic functions of the inverse of the network nodal admittance matrix. However, similar to the node voltages in (5.19) and (5.10), network transimpedances are rarely calculated manually through an actual delineation and inversion of the nodal admittance matrix. Instead, they usually derive from a straightforward analysis of the considered network, subject to the proviso that all excitations applied to the subject network, save for the single test current source, are reduced to zero. For example, in the abstraction shown in Figure 5.7, the transimpedance, Z_{ij}, from any port *j* to any port *i* is

$$Z_{ij} \stackrel{\Delta}{=} \left. \frac{V_{\text{test}}}{I_{\text{test}}} \right|_{\text{all independent sources}=0} \tag{5.22}$$

For the case of $j = i$, this transimpedance becomes the effective impedance seen at port *i* by the test current source. In view of the preceding discussion, and the node pairs indicated in Figure 5.7, the transimpedance (or impedance) quantity that derives from (5.22) is identical to the matrix relationship

$$Z_{ij} = \left(\mathbf{A}_{cd}^{\mathrm{T}} \mathbf{Y}^{-1} \mathbf{A}_{ab}\right) \tag{5.23}$$

The last result highlights the fact that all network transimpedances are directly related to the inverse of the nodal admittance matrix. Hence, these transimpedances are inversely proportional to the determinant, $\Delta Y(s)$, of the admittance matrix, **Y**. It follows that the poles of all transimpedances and effective port impedances are the roots of the characteristic polynomial

$$\det(\mathbf{Y}) \stackrel{\Delta}{=} \Delta Y(s) = 0 \tag{5.24}$$

Note from (5.7), (5.17), and (5.20) that the voltage gain of the considered linear network is

$$\frac{V_L}{V_S} = \frac{Z_{LS}}{Z_S} \tag{5.25}$$

Thus, if the source impedance in Figure 5.5 is a real number, $Z_S = R_S$, the roots of (5.24) also comprise the poles of the voltage transfer function and, indeed, of the linear network.

Kron's Formula

Assume now that the network depicted in Figure 5.5 has been analyzed in the sense that all network node voltages developed in response to the signal source have been determined. Assume further that subsequent to this analysis, an impedance, Z_k, appended to nodes m and p, as shown in Figure 5.8. In addition to causing a current, I, to flow into node m and out of node p, this additional branch element is likely to perturb the values of all of the originally computed circuit node and circuit branch voltages. The matrix, $\mathbf{E}'$, of new node voltages can be evaluated for the modified topology in Figure 5.8 by determining the new nodal admittance matrix $\mathbf{Y}'$, and the reapplying (5.9). The tedium associated with a second network analysis, along with the inefficiency of discarding the results of a study performed on a network whose topology differs only modestly from that of the original configuration, can be circumvented through the use of *Kron's theorem*. As illuminated next, this theorem derives from a methodical application of such classical concepts as the theories of superposition, substitution, and Thévenin. In addition to providing a computationally efficient mechanism for determining $\mathbf{E}'$, Kron's technique allows for a direct comparison of $\mathbf{E}'$ to the matrix, $\mathbf{E}$, of original node voltages. It therefore allows for a convenient response sensitivity analysis with respect to the appended branch element.

It is appropriate to interject that the problem postulated previously possesses more than mere academic interest. It is, in fact, a problem that is commonly encountered, for example, in the analysis of electronic circuits. In order to linearize these circuits around specified quiescent operating points, it is necessary to supplant the utilized active devices by small signal equivalent circuits. Such models are invariably simplified, often through the tacit neglect of presumably noncritical branch elements, to mitigate analytical complexity and tedium. Thus, while the circuit properly identified for investigation might be of the topological form appearing in Figure 5.8, the circuit actually subjected to manual circuit analysis is likely the reduced structure depicted in Figure 5.5; that is, the ostensibly noncritical impedance, Z_k is removed in the interest of analytical tractability. Questions naturally arise in regard to the degree of error incurred by the invoked circuit simplification. Kron's method, as developed next, answers these questions in terms of the results already deduced for the approximate network and without requiring explicit analytic results for the "exact" network.

The process of evaluating the perturbation on network node voltages incurred by the action of shunting nodes m and p in the circuit of Figure 5.5 by the impedance Z_k begins by determining the Thévenin equivalent circuit that drives the appended branch. To this end, Z_k is removed in the diagram of Figure 5.8, thereby collapsing the network to Figure 5.5(a). The relevant Thévenin voltage, V_{th}, at the node pair, m-p, is, in fact, V_k, as defined by (5.18). Recalling (5.21), this voltage is

$$V_{\text{th}} \equiv V_k = \left(\mathbf{A}_{mp}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{A}_{12}\right)I_S = Z_{kS}I_S \tag{5.26}$$

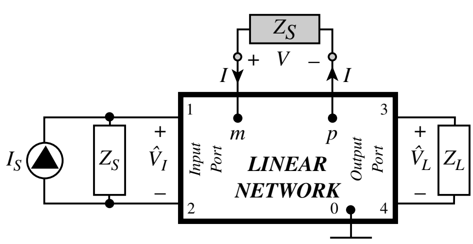

FIGURE 5.8 The inclusion of an impedance, Z_k, between nodes m And p, subsequent to the analysis of the network in Figure 5.5.

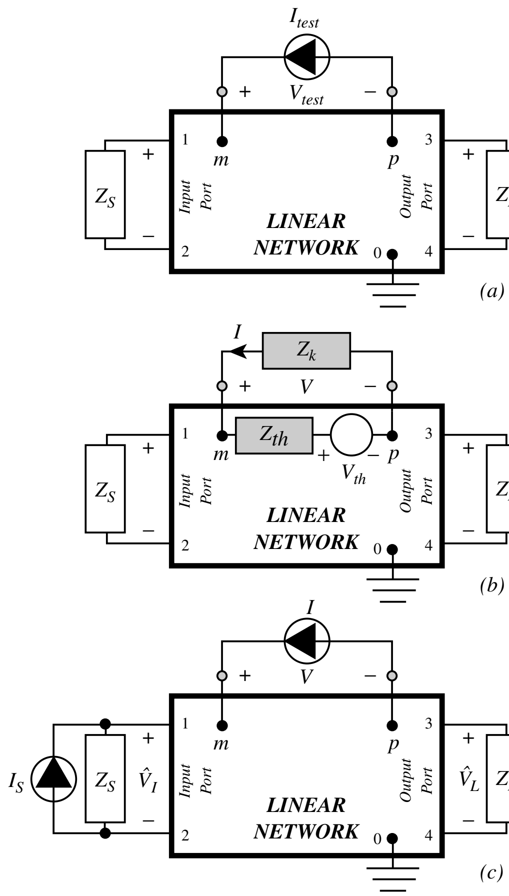

FIGURE 5.9 (a) Circuit diagram for evaluating the Thévenin impedance seen by the appended impedance Z_k. (b) Circuit diagram used to compute the current, I, conducted by Z_k. (c) The application of the substitution theorem with respect to Z_k.

The corresponding Thévenin impedance, Z_{th}, derives from a study of the test configuration of Figure 5.9, in which the independent source current, I_S, is nulled, the impedance, Z_k, in Figure 5.8 is replaced by a test current of value I_{test}, and the ratio of the resultant port voltage, V_{test}, to I_{test} is understood to be the desired Thévenin impedance. For this configuration, the network nodal admittance matrix, Y, is unchanged, but the independent network current vector, J, becomes $A_{mp}I_{\text{test}}$. Thus, the resultant n-vector, E″, of nodal voltages is

$$\mathbf{E}'' = \mathbf{Y}^{-1}\mathbf{A}_{mp}I_{\text{test}} \tag{5.27}$$

and by (5.8), the voltage, V_{test}, is

$$V_{\text{test}} = \left(\mathbf{A}_{mp}^{\text{T}}\mathbf{Y}^{-1}\mathbf{A}_{mp}\right)I_{\text{test}} \tag{5.28}$$

It follows that the requisite Thévenin impedance, Z_{th}, is

$$Z_{\text{th}} = \frac{V_{\text{test}}}{I_{\text{test}}} = \left(\mathbf{A}_{mp}^{\text{T}}\mathbf{Y}^{-1}\mathbf{A}_{mp}\right) \tag{5.29}$$

Insofar as the appended impedance, Z_k, is concerned, the network in Figure 5.8 behaves in accordance with the circuit abstraction of Figure 5.9. The current, I, conducted by Z_k is, without approximation,

$$I = -\frac{V_{\text{th}}}{Z_{\text{th}} + Z_k} = -\left(\frac{Z_{kS}}{Z_{\text{th}} + Z_k}\right) I_S \tag{5.30}$$

where (5.26) has been used, and Z_{th} is understood to be given by (5.29). However, by the substitution theorem, the impedance, Z_k in Figure 5.8 can be supplanted by an independent current source of value I, as suggested in Figure 5.9(c). Specifically, this substitution of the impedance of interest with a current source with a value that is dictated by (5.30) guarantees that the n-vector of node voltages for the modified circuit in Figure 5.9(c) is identical to the n-vector, E', of node voltages for the topology given in Figure 5.8.

The circuit of Figure 5.9(c) now has two independent excitations: the original current sources, I_S, and the current, I, substituted for the appended impedance, Z_k. Accordingly, the current source vector for the subject circuit superimposes two current components and is given by

$$\mathbf{J} = \mathbf{A}_{12} I_S + \mathbf{A}_{mp} I = \mathbf{A}_{12} I_S - \mathbf{A}_{mp}\left(\frac{Z_{kS}}{Z_{\text{th}} + Z_k}\right) I_S \tag{5.31}$$

The corresponding vector of node voltages is, by (5.9),

$$\mathbf{E}' = \mathbf{Y}^{-1}\left[\mathbf{A}_{12} - \mathbf{A}_{mp}\left(\frac{Z_{kS}}{Z_{\text{th}} + Z_k}\right)\right] I_S \tag{5.32}$$

If analytical attention focuses on the general output voltage, $\hat{V}_{ij}$, developed between nodes i and j in the circuit of Figure 5.8,

$$\hat{V}_{ij} = \mathbf{A}_{ij}^T \mathbf{E}' \tag{5.33}$$

where

$$\hat{V}_{ij} = \left[\left(\mathbf{A}_{ij}^{\mathrm{T}} \mathbf{Y}^{-1} \mathbf{A}_{12}\right) - \left(\mathbf{A}_{ij}^{\mathrm{T}} \mathbf{Y}^{-1} \mathbf{A}_{mp}\right)\left(\frac{Z_{kS}}{Z_{\text{th}}} + Z_k\right)\right] I_S \tag{5.34}$$

The result in (5.34) is one of many possible versions of *Kron's formula*. It states that when an impedance, Z_k, is appended between nodes m and p of a linear network whose nodal admittance matrix is **Y**, the perturbed voltage established between any two nodes, i and j, can be determined as a function of the parameters indigenous to the original network (prior to the inclusion of Z_k). In particular, the evaluation is executed on the original network (with Z_k absent) and exploits the original nodal admittance matrix, **Y**, the original transimpedance, Z_{kS}, between the input port and the port to which Z_k is ultimately appended, and the Thévenin impedance, Z_{th}, is observed when looking into the terminal pair to which Z_k is connected.

Engineering Application of Kron's Formula

The engineering utility of Kron's formula, (5.34), is best demonstrated by examining the voltage transfer function of the network in Figure 5.8 in terms of the companion gain for the network depicted in Figure 5.5(a). Using (5.7) and noting that the perturbed output voltage is developed from node 3 to node 4, the perturbed voltage gain, $\hat{A}_v$, is

$$\hat{A}_v \triangleq \frac{\hat{V}_L}{V_S} = \frac{\mathbf{A}_{34}^{\mathbf{T}} \mathbf{Y}^{-1} \mathbf{A}_{12}}{Z_S} - \left(\frac{\mathbf{A}_{34}^{\mathbf{T}} \mathbf{Y}^{-1} \mathbf{A}_{mp}}{Z_S}\right)\left(\frac{Z_{kS}}{Z_{\text{th}} + Z_k}\right) \tag{5.35}$$

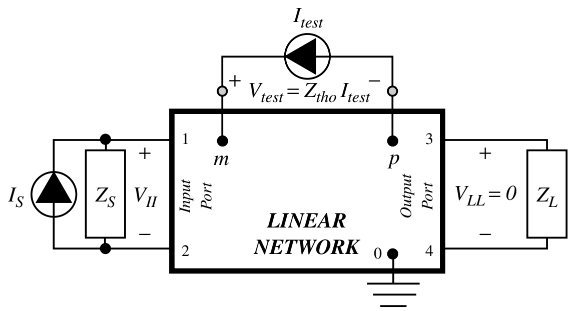

FIGURE 5.10 Network diagram pertinent to the computation of the null Thévenin impedance seen by the impedance appended to the node pair, *m-p*.

The first matrix product on the right-hand side of this relationship represents the transimpedance, Z_{LS}, from the input port to the output port of the original network, as given by (5.20). Moreover, the resultant impedance ratio, Z_{LS}/Z_S, is the voltage gain, A_v, of the original ($Z_k = \infty$) network, as delineated in (5.25). The second matrix product symbolizes the transimpedance, Z_{LK}, from the port to which the appended impedance, Z_k, is connected to the output port; that is,

$$Z_{Lk} = \mathbf{A}_{34}^{\mathrm{T}} \mathbf{Y}^{-1} \mathbf{A}_{mp} \tag{5.36}$$

Assuming $A_v \neq 0$, (5.35) can then be reduced to

$$\hat{A}_v = A_v \left[1 - \left(\frac{Z_{Lk}}{Z_{LS}} \right) \left(\frac{Z_{kS}}{Z_{\mathrm{th}} + Z_k} \right) \right] \tag{5.37}$$

This result expresses the perturbed voltage gain as a function of the original voltage gain, A_v, the input to output transimpedance, Z_{LS}, the transimpedance, Z_{kS}, from the input port to the port at which Z_k is appended, and Z_{LK}, the transimpedance from the port to which Z_k is incident to the output port. Observe that when the appended impedance is infinitely large, the perturbed gain reduces to the original voltage gain, as expected.

In an actual analytical situation, however, all of the transimpedances indicated in (5.37) need not be calculated. In order to demonstrate this contention, rewrite (5.37) in the form

$$\hat{A}_v = A_v \left[\frac{1 + Y_k \left(Z_{\mathrm{th}} - \left(Z_{Lk} Z_{kS} / Z_{LS} \right) \right)}{1 + Y_k Z_{\mathrm{th}}} \right] \tag{5.38}$$

where

$$Y_k = \frac{1}{Z_k} \tag{5.39}$$

is the admittance of the appended impedance, Z_k. Now consider the test structure of Figure 5.10(a), which is the modified circuit shown in Figure 5.8, but with the appended branch supplanted by a test current source, I_{test}. With two sources, I_S and I_{test}, activating the network, superposition yields a resultant output port voltage, V_{LL}, of

$$V_{LL} = Z_{LS} I_S + Z_{LK} I_{\mathrm{test}} \tag{5.40}$$

and a test port voltage, V_{test}, of

$$V_{\text{test}} = Z_{kS}I_S - Z_{kk}I_{\text{test}} \tag{5.41}$$

For $I_S = 0$, the network in Figure 5.10(a) reduces to the configuration in Figure 5.9(a), and (5.41) delivers $V_{\text{test}}/I_{\text{test}} = Z_{kk}$. It follows that the impedance parameter, Z_{kk}, is the Thévenin impedance seen by Z_k, as determined in conjunction with an analytical consideration of Figure 5.5(a); that is, (5.41) is equivalent to the expression

$$V_{\text{test}} = Z_{kS}I_S + Z_{\text{th}}I_{\text{test}} \tag{5.42}$$

Consider the case, suggested in Figure 5.10, in which the output port voltage, V_{LL}, is constrained to zero for any and all values of the load impedance, Z_L. From (5.40), this case requires a source excitation that satisfies

$$I_S = -\left(\frac{Z_{Lk}}{Z_{LS}}\right)I_{\text{test}} \tag{5.43}$$

If this result is substituted into (5.42), the ratio, $V_{\text{test}}/I_{\text{test}}$, is found to be

$$\frac{V_{\text{test}}}{I_{\text{test}}} = Z_{th} - \frac{Z_{Lk}Z_{kS}}{Z_{LS}} \tag{5.44}$$

which mirrors the parenthesized numerator term on the right-hand side of (5.38). The ratio in (5.44) might rightfully be termed a *null Thévenin impedance*, Z_{tho}, seen by Z_k, in the sense that it is indeed the Thévenin impedance witnessed by Z_k, but under the special circumstance of a nonzero source excitation selected to *null* the output response variable of the network undergoing investigation. Thus, in Figure 5.10,

$$\left.\frac{V_{\text{test}}}{\text{I}_{\text{test}}}\right|_{\substack{\text{Source}\neq 0\\ \text{Output}=0}} \triangleq Z_{\text{tho}} = Z_{\text{th}} - \frac{Z_{LK}Z_{kS}}{Z_{LS}} \tag{5.45}$$

Equation (5.38) now reduces to the simpler result

$$\hat{A}_v = A_v\left(\frac{1 - Y_kZ_{\text{tho}}}{1 + Y_kZ_{\text{th}}}\right) = A_v\left(\frac{1 + \left(Z_{\text{tho}}/Z_k\right)}{1 + \left(Z_{\text{th}}/Z_k\right)}\right) \tag{5.46}$$

Equation (5.46) is both computationally useful and philosophically important. From a computational viewpoint, it allows for an efficient evaluation of the voltage transfer function of a linear network, perturbed by the addition of an impedance element between two extant nodes of the network, in terms of the voltage gain of the original, unperturbed circuit. As expected, this original voltage gain, A_v, is the voltage gain of the perturbed network for the special case of a perturbing impedance where the admittance is zero (or whose impedance value is infinitely large). Only two other parameters are required to complete the evaluation of the perturbed gain. The first is the Thévenin impedance, Z_{th}, seen by the appended impedance element. This Thévenin impedance is calculated traditionally by nulling all independent sources applied to the subject network. The second parameter is the null Thévenin impedance, Z_{tho}, which is the value of Z_{th} for the special circumstance of a test current source and independent source excitations selected to constrain the output response variable to zero. Once Z_{tho} and Z_{th} are determined, the degree to which the voltage transfer function is dependent on the appended impedance is easily determined. For example, the per-unit change in gain owing to the addition of Z_k between nodes m and p in the network of Figure 5.8 is

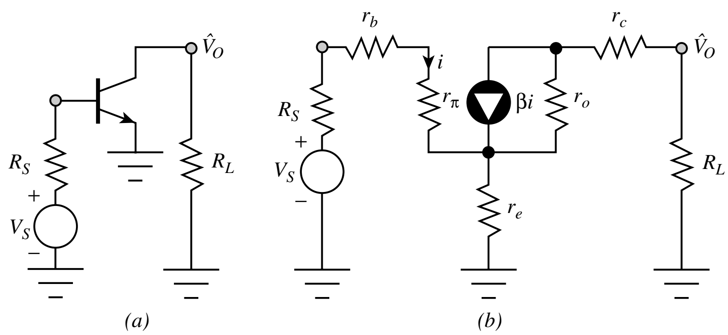

FIGURE 5.11 (a) Simplified schematic diagram of a common emitter amplifier. (b) The small signal equivalent circuit of the common emitter amplifier.

$$\frac{\Delta A_v}{A_v} = \frac{\hat{A}_v - A_v}{A_v} = \frac{Z_{\text{tho}} - Z_{\text{th}}}{Z_k + Z_{\text{th}}} \tag{5.47}$$

it is important to note that the transfer function sensitivity implied by (5.47) imposes no *a priori* restrictions on the value of Z_k. In particular, Z_k can assume any value from that of a short circuit to that of the opposite extreme of an open circuit.

From a philosophical perspective, when analytical attention focuses explicitly on feedback circuits, (5.46) can be derived from signal flow theory [10, 11] and is actually Bode's classical gain equation [12]. In the context of Bode's theory Y_k is referred to as a *reference parameter*, or a *critical parameter*, of the feedback circuit. The product, $Y_k Z_{\text{th}}$, is termed the *return ratio with respect to the critical parameter*, while the product $Y_k Z_{\text{tho}}$, is identified as the *null return ratio with respect to the cricital parameter*. Finally, when (5.46) is applied as Bode's equation, the transfer function, A_v is termed the *null transfer function*, in the sense that it is the actual transfer function of the network at hand, under the special case of a critical parameter constrained to zero.

Example 5.1. In an attempt to demonstrate the engineering utility of the foregoing theoretical disclosures, consider the problem of determining the voltage gain of the common emitter amplifier — a schematic diagram is offered in Figure 5.11(a). Without detracting from the primary intent of this example, the schematic diagram at hand has been simplified in that requisite biasing subcircuitry is not shown. Assuming that the bipolar junction transistor embedded in the amplifier operates in its linear regime, the pertinent small signal equivalent circuit is the topology depicted in Figure 5.11(b).

Assume that the amplifier source resistance, R_S, is 600 Ω and that the load resistance, R_K, is 10 kΩ. Assume further that the model parameters for the transistor are as follows: r_b (internal emitter resistance) = 2.5 Ω, r_0 (forward early resistance) = 18 kΩ, β (forward short circuit current gain) = 90, and r_c (internal collector resistance) = 70 Ω. Determine the voltage gain of the amplifier and the effect exerted on the gain by neglecting the Early resistance, R_0.

Solution.

1. Analytical simplicity traditionally dictates the tacit neglect of the forward Early resistance, r_0. This commonly invoked approximation reduces the model given in Figure 5.11(b) to the equivalent circuit in Figure 5.12(a). By inspection of the latter diagram, the approximate gain of the common emitter voltage is

$$A_v = \frac{V_O}{V_S} = -\frac{\beta R_L}{R_S + r_b + r_\pi + (\beta+1) r_e} = -388.3 \text{ v/v}$$

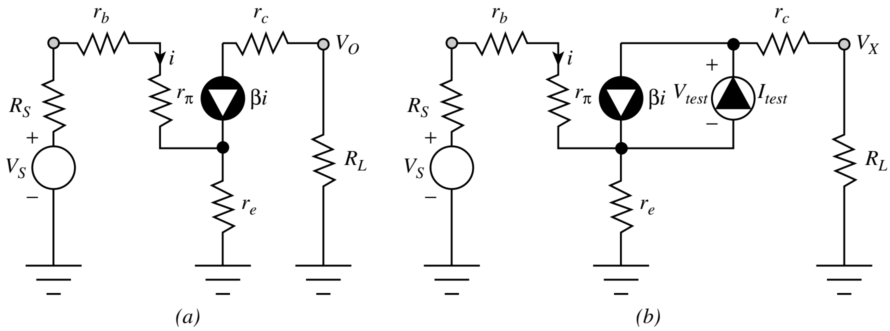

FIGURE 5.12 (a) The approximate small signal equivalent circuit of the common emitter amplifier in Figure 5.11(a). The approximation entails the tacit neglect of the forward Early resistance, r_0. (b) The test equivalent circuit used to compute the Thévenin and the null Thévenin resistances seen by r_0 in (a).

2. In order to determine the impact that r_0 has on this voltage gain, r_0 is removed from the equivalent circuit and replaced by a test current source, I_{test}, as depicted in Figure 5.12(b). With the independent input voltage, V_S, set to zero, the Thévenin resistance seen by r_0, which is the ratio, $V_{\text{test}}/I_{\text{test}}$, is easily shown to be

$$R_{\text{th}} = r_e \left\| \left(\frac{R_S + r_b + r_n}{\beta + 1} \right) + \frac{r_c + R_L}{1 + \dfrac{\beta r_e}{R_S + r_b + r_\pi + r_e}} \right. = 9.10 \text{ k}\Omega$$

On the other hand, if V_X is constrained to zero, no current flows through the load resistance branch, and therefore, I_{test} is necessarily βi. This condition gives a null Thévenin resistance of

$$R_{\text{tho}} = -\frac{r_e}{\beta} = -27.78 \times 10^{-3}\,\Omega$$

3. With $A_v = -388.3$ v/v, $Z_k = r_0 = 18$ kΩ, $Z_{\text{th}} = R_{\text{th}} = 9.10$ kΩ, and $Z_{\text{tho}} = R_{\text{tho}} = -\,27.78 \times 10^{-3\prime}$ Ω, (5.46) produces a corrected voltage gain of

$$\hat{A}_v = A_v \left(\frac{1 + \dfrac{R_{\text{tho}}}{r_0}}{1 + \dfrac{R_{\text{th}}}{r_0}} \right) = -258.0 \text{ v/v}$$

From (5.47), the presence of r_0, as opposed to its absence, decreases the voltage gain of the subject amplifier by almost 34%.

Example 5.2. As a second example, consider the series-shunt feedback amplifier whose schematic diagram, neglecting requisite biasing circuitry, appears in Figure 5.13(a). The analysis of this circuit is simplified by the removal of the connection of the feedback resistance, R_F, at the emitter of transistor $Q1$, as shown in Figure 5.13(b). If the voltage gain of the simplified topology is denoted as A_v, the voltage gain of the *closed loop* configuration in Figure 5.13(a) derives from (5.46), provided that the impedance, Z_k, between the indicated node pair, *m-p*, is taken as a short circuit; that is, $Z_k = 0$.

Assume that the amplifier source resistance, R_S, is 300 Ω, the load resistance, R_L, is 3.5 kΩ, the feedback resistance, R_F, is 1.5 kΩ, and the emitter degeneration resistance, R_{EE}, is 100 Ω. The transistor model invoked for small signal analysis is identical to that used in the preceding example, save for the proviso

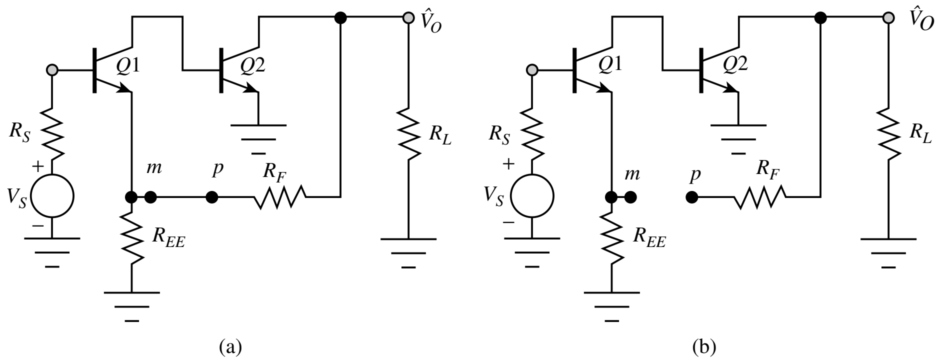

FIGURE 5.13 (a) Simplified schematic diagram of a series-shunt feedback bipolar junction transistor amplifier. The biasing subcircuitry is not shown. (b) The amplifier in (a), but with the feedback resistance connection to the emitter of transistor $Q1$ removed.

that the Early resistance, r_0, is ignored herewith. Both transistors are presumed to have identical small signal model parameters as follows: $r_b = 90\ \Omega$, $r_\pi = 1.4\ \text{k}\Omega$, $r_e = 2.5\ \Omega$, $\beta = 90$, and $r_c = 70\ \Omega$. Use Kron's theorem to determine the voltage gain of the closed loop series-shunt feedback amplifier.

Solution.

1. The voltage gain of the pertinent equivalent circuit in Figure 5.14 is straightforwardly derived as

$$A_v = \frac{V_O}{V_S} = \frac{\beta^2 R_L}{R_S + r_b + r_\pi + (\beta+1)(r_e + R_{EE})} = 2550\ \text{v/v}$$

   Observe that the feedback resistance, R_F, does not enter into this calculation because of its disconnection at the emitter of transistor $Q1$. Furthermore, the internal collector resistance, r_c, is inconsequential for both transistor stages because the neglect of the forward Early resistance, r_0, places r_c in series with a controlled current source.

2. The Thévenin resistance, R_{th}, seen by the ultimately appended short circuit between nodes m and p is now calculated through use of the model in Figure 5.14(b). For this calculation, the signal voltage, V_S, is reduced to zero. With $V_S = 0$,

$$\frac{i_1}{i_{\text{test}}} = -\frac{R_{EE}}{R_S + r_b + r_\pi + (\beta+1)(r_e + R_{EE})}$$

   Noting that $i_2 = -\beta i_1$, KVL yields

$$V_{\text{test}} = (R_{EE} + R_L + R_F)I_{\text{test}} + \left[(\beta+1)R_{EE} - \beta^2 R_L\right]i_1$$

   Using the preceding result, introducing the resistance variable, R_X, such that

$$R_X \triangleq r_e + \frac{R_S + r_b + r_\pi}{\beta+1} = 22.17\Omega$$

   and letting

$$\alpha \triangleq \frac{\beta}{\beta+1} = 0.989$$

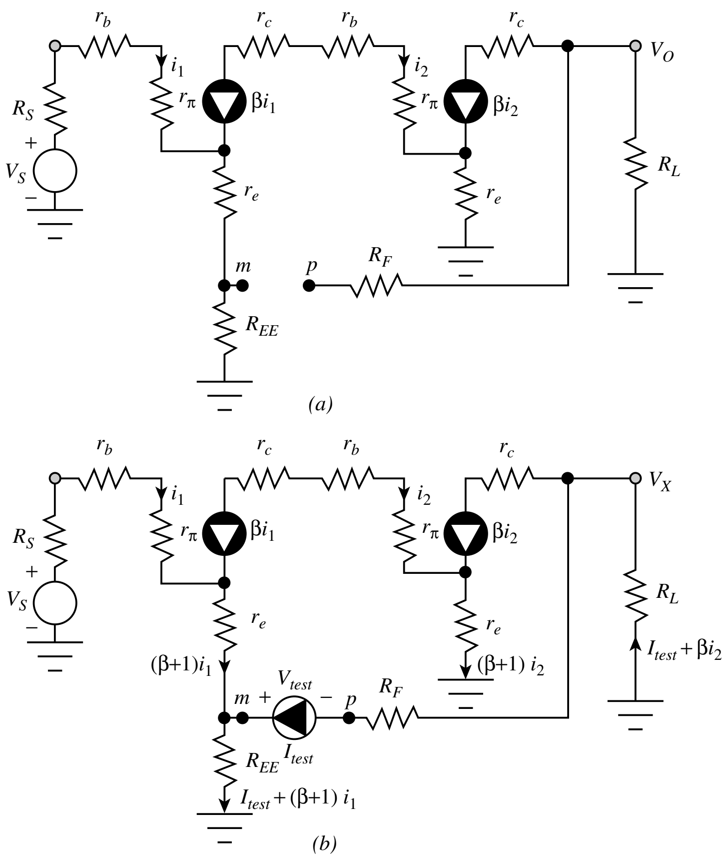

FIGURE 5.14 (a) Small signal equivalent circuit of the feedback amplifier in Figure 5.13(b). This circuit is used to compute the voltage gain with the feedback resistance disconnected at the emitter of transistor $Q1$. (b) The small signal model used to compute the Thévenin and the null Thévenin resistances seen by the short circuit that is ultimately appended to the node pair, m-p, in Figure 5.13(b).

R_{th}, which is the ratio $V_{\text{test}}/I_{\text{test}}$, is found to be

$$R_{\text{th}} = R_F + \left(R_{EE} \| R_X\right)\left(1 + \frac{\alpha\beta R_{EE}}{R_{EE} + R_X}\right) R_L = 260.0 \ \ K\Omega$$

3. For the evaluation of null Thévenin resistance, R_{tho}, the output voltage variable, V_X, in Figure 5.14(b) is nulled, thereby forcing the current relationship, $I_{\text{test}} = -\beta i_2 = +\beta^2 i_1$. Accordingly,

$$R_{\text{tho}} = R_F + \left(1 + \frac{1}{\alpha\beta}\right) R_{EE} = 1601\Omega$$

4. With $Z_k = 0$, $Z_{\text{th}} = R_{\text{th}} = 260\ \text{k}\Omega$, and $Z_{\text{tho}} = R_{\text{tho}} = 1601\ \Omega$, (5.46) provides, after reconnection of the feedback element, an amplifier gain of

$$\hat{A}_v = A_v \left(\frac{R_{\text{tho}}}{R_{\text{th}}}\right) = 15.70 \ \ \text{V/V}$$

It is interesting to note that, if the transistors utilized in the feedback amplifier have very large β, which is tantamount to very small R_X and $a \approx 1$, the voltage gain with the feedback element disconnected is

$$a_v \approx \frac{\beta R_L}{R_{EE}}$$

Moreover,

$$R_{\text{th}} \approx \beta R_L$$

and

$$R_{\text{tho}} \approx R_F + R_{EE}$$

It follows from (5.46) that the approximate voltage gain, subsequent to the reconnection of the feedback resistance, R_F, to the emitter of transistor $Q1$ is

$$\hat{A}_v = A_v\left(\frac{R_{\text{tho}}}{R_{\text{th}}}\right) \approx \left(\frac{\beta R_L}{R_{EE}}\right)\left(\frac{R_F + R_{EE}}{\beta R_L}\right) = 1 + \frac{R_F}{R_{EE}} = 16 \text{ V/V}$$

which is within 2% of the accurately estimated voltage gain.

Generalization of Kron's Formula

The Kron–Bode equation in (5.46) was derived expressly for investigating the voltage transfer function of a linear network to which an impedance element is appended between two network nodes. This equation also can be adapted to the problem of determining the explicit dependence of any type of transfer relationship on any parameter within any linear network.

To this end, consider any linear network, such as the generalization shown in Figure 5.15, whose, load impedance is Z_L and whose source impedance is Z_S. Identify a *critical network parameter*, say P, to which the dependence on, and sensitivity to, the overall transfer performance of the network undergoing study is of particular interest. This parameter can be, for example, a circuit branch impedance or an active element gain factor where numerical values cannot be determined accurately or controlled adequately in view of potentially unacceptable manufacturing tolerances or device fabrication uncertainties. Let the transfer function of interest be

$$H(P, Z_S, Z_L) = \frac{X_R(s)}{X_S(s)} \tag{5.48}$$

where $X_R(s)$ denotes the transform of the voltage or current response variable, and $X_S(s)$ is the transform of the voltage or current input variable. The functional notation, $H(P, Z_S, Z_L)$, underscores the observation

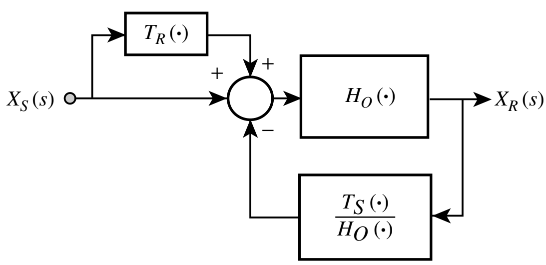

FIGURE 5.15 Generalized block diagram nodal of the I-O transfer characteristics of a linear network.

that the transfer function of the liner network is likely to be dependent on the critical parameter, P, the source impedance, Z_S, and the load impedance, Z_L. The corresponding extension of the Kron–Bode relationship is

$$H\left(P, Z_S, Z_L\right) = \frac{X_R(s)}{X_S(s)} = H\left(0, Z_S, Z_L\right)\left[\frac{1 + PQ_R\left(Z_S, Z_L\right)}{1 + PQ_S\left(Z_S, Z_L\right)}\right] \tag{5.49}$$

where $H(0, Z_S, Z_L)$, termed the *null gain or zero parameter gain,* signifies the value of the network transfer function, $H(P, Z_S, Z_L)$, when P is set to zero. This null gain must be finite and nonzero. With reference to the appended impedance formulation in (5.46), observe that the critical parameter, P, is Y_k, the admittance of the appended impedance element, while $H(0, Z_S, Z_L)$ is the gain, A_v, of the network, under the condition of an absent impedance element ($Y_k = 0$).

The product, PQ_S (Z_S, Z_L), is termed the *return ratio with respect to parameter P*, T_s (P, Z_S, Z_L), and the product, PQ_R (Z_S, Z_L), is referred to as the *null return ratio with respect to P*, T_R (P, Z_S, Z_L); that is,

$$T_R\left(P, Z_S, Z_L\right) \triangleq PQ_R\left(Z_S, Z_L\right) \tag{5.50a}$$

$$T_S\left(P, Z_S, Z_L\right) \triangleq PQ_S\left(Z_S, Z_L\right) \tag{5.50b}$$

It is to be understood that both Q_S (Z_S, Z_L) and Q_R (Z_S, Z_L) are independent of the critical parameter, P. With reference once again to (5.46), note that Q_S (Z_S, Z_L) is the Thévenin impedance seen by the appended admittance, Y_k, while Q_S (Z_S, Z_L) is the null Thévenin impedance facing Y_K.

Equation (5.49) can now be rewritten as

$$H\left(P, Z_S, Z_L\right) = \frac{X_R(s)}{X_S(s)} = H\left(0, Z_S, Z_L\right)\left[\frac{1 + T_R\left(P, Z_S, Z_L\right)}{1 + T_S\left(P, Z_S, Z_L\right)}\right] \tag{5.51}$$

Alternatively,

$$H\left(P, Z_S, Z_L\right) = \frac{X_R(s)}{X_S(s)} = H\left(0, Z_S, Z_L\right)\left[\frac{F_R\left(P, Z_S, Z_L\right)}{F_S\left(P, Z_S, Z_L\right)}\right] \tag{5.52}$$

where

$$F_S\left(P, Z_S, Z_L\right) \triangleq 1 + T_S\left(P, Z_S, Z_L\right) \tag{5.53a}$$

$$F_R\left(P, Z_S Z_L\right) \triangleq 1 + T_R\left(P, Z_S, Z_L\right) \tag{5.53b}$$

respectively, denote the *return difference with respect to P* and the *null return difference with respect to P*.

An initial appreciation of the engineering significance of the zero parameter gain, H $(0, Z_S, Z_L) \equiv H_0$ $(\cdot)$, the return ratio, T_S $(P, Z_S, Z_L) \equiv T_S$ $(\cdot)$, and the null return ratio, T_R $(P, Z_S, Z_L) \equiv T_R$ $(\cdot)$, is gleaned by using (5.51) to write

$$X_R(s) = H_0(\cdot)\left[1 + T_R(\cdot)\right]X_S(s) - T_S(\cdot)X_R(s) \tag{5.54}$$

In view of the generality of the Kron–Bode formula, this algebraic manipulation of (5.51) implies that the dynamical input/output transfer relationship of all linear networks can be symbolically represented by the block diagram offered in Figure 5.15. This block diagram makes clear that because T_S $(\cdot)$ and T_R $(\cdot)$ are zero for $P = 0$, H_0 $(\cdot)$ is the gain afforded by the network as a result of input–output electrical paths that exclude the parameter, P. The diagram also suggests that $T_S(\cdot)$ is a measure of the amount of feedback incurred by parameter P around that part of the circuit that excludes parameter P. Finally, the diagram at hand implies that $T_R(\cdot)$ is a measure of the amount of *feedforward* incurred by parameter P. In particular, if feedback is removed, two paths remain for the transmission of a signal from the input port to the output port of a linear network. One of these paths, the transmittance of which is measured by $H_0(\cdot)$, is direct and entails the processing of the input signal by that part of the circuit that excludes P. The other nonfeedback path has an effective transmittance of $T_R(\cdot)\,H_0(\cdot)$. The latter path is the feedforward path in the sense that signal is processed through a signal path that is divorced from feedback and is not a result of direct source coupling through the topological part of the network that excludes parameter P.

Return Ratio Calculations

In the generalized transfer relationship of (5.49), the critical parameter, P, can assume one of only six possible forms: circuit branch admittance, circuit branch impedance, transimpedance, transadmittance, gain associated with a current-controlled current source (CCCS), and gain associated with a voltage-controlled voltage source (VCVS) [13]. The methodology underlying the computation of the return ratio and the null return ratio with respect to each of these critical parameter possibilities is given below. The case of $P = Y_k$, a circuit branch admittance, was investigated in the context of Kron's partitioning theorem. Nevertheless, it is reinvestigated next for the purpose of establishing an analytical common denominator for return ratio calculations with respect to the five other reference parameter possibilities.

Circuit Branch Admittance

Consider the network abstraction of Figure 5.16(a), which identifies a branch admittance, Y_k, as a critical parameter for analysis; that is, $P = Y_k$ in (5.49). The input excitation can be either a voltage source, or a current source and is therefore indicated as a general transformed input variable, $X_S(s)$. Similarly, the

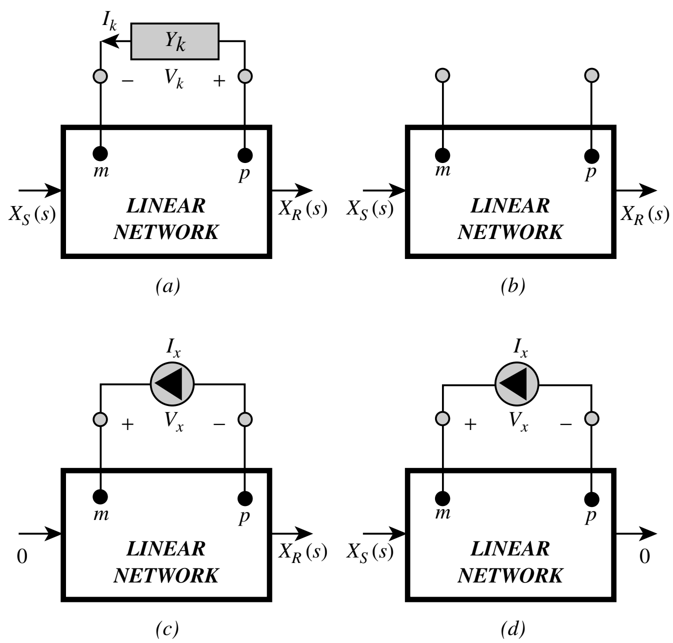

FIGURE 5.16 (a) Linear circuit for which the identified critical parameter is a branch admittance, Y_k. (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain $H(0, Z_S, Z_L)$. (c) The ratio, V_x/I_x, is the function $Q_S(Z_S, Z_L)$, in (5.49). (d) The ratio, V_x/I_x, is the function, $Q_R(Z_S, Z_L)$, in (5.49).

output or response variable can be either a voltage or a current, thereby encouraging the generalized transformed response notation, $X_R(s)$. The source and load impedances (or admittances) are absorbed into the network. In order to evaluate the zero parameter gain, $H(0, Z_S, Z_L)$, Y_k is set to zero by removing it from the network. An analysis is then conducted to determine the ratio $X_R(s)/X_S(s)$, of output to input variables, as suggested in Figure (5.16(b)).

As demonstrated for the case of $P = Y_k$, a circuit branch admittance, the function, $Q_S(Z_S, Z_L)$, in (5.49) is the Thévenin impedance, Z_{th}, facing Y_k. This impedance is computed by determining the ratio, V_x/I_x, with the signal source, $X_S(s)$, nulled, as indicated in Figure 5.16(c). Note in Figure 5.16(a), that the volt–ampere relationship of the branch housing Y_k is $I_k = Y_k V_k$, where the direction of the branch current, I_k, coincides with the direction of the test current source, I_x, used in the determination of Z_{th}. A comparison Figures 5.16(c) and 5.16(a) alludes to the methodology of replacing the admittance branch by a source of excitation (a current source) where the electrical nature is identical to the *dependent* electrical variable (a current, I_k) of the branch volt–ampere characteristic. Note that the polarity of the voltage, V_x, used in the determination of the test ratio, V_x/I_x, is opposite to that of the original branch voltage V_k. This is to say that although I_k and V_k are in associated reference polarity in Figure 5.16(a), I_x and V_x in the test cell of Figure 5.16(c) are in disassociated polarity.

The computation of the function, $Q_R(Z_S, Z_L)$, in (5.49) mirrors the computation of $Q_S(Z_S, Z_L)$, except for the fact that instead of setting the source excitation to zero, the output response, $X_R(s)$, is nulled. The source excitation, $X_S(s)$, remains at some computationally unimportant nonzero value, such that its effects, when superimposed over those of the test current, I_x, forces $X_R(s)$ to zero. The situation at hand is diagrammed in Figure 5.16(d).

Example 5.3. Return to the series-shunt feedback amplifier of Figure 5.29(a). Evaluate the voltage gain of the circuit, but, take the conductance, G_F, of the feedback resistance, R_F, as the critical parameter. The circuit and device model parameters remain the same as in Example 5.2: $R_S = 300\ \Omega$, $R_L = 3.5\ \text{k}\Omega$, $R_F = 1.5\ \text{k}\Omega$, $R_{EE} = 100\ \Omega$, $r_b = 90\ \Omega$, $r_\pi = 1.4\ \text{k}\Omega$, $r_e = 2.5\ \Omega$, $\beta = 90$, and $r_c = 70\ \Omega$.

Solution.

1. The zero parameter voltage gain, A_{vo}, of the subject amplifier is the voltage gain of the circuit with $G_F = 0$. But $G_F = 0$ amounts to a removal of the feedback resistance, R_F. Such removal is electrically equivalent to open circuiting the indicated node pair, *m-p*, as diagrammed in the small signal model of Figure 5.14(a). Thus, A_{vo} is identical to the gain, computed in Step (1) of Example 5.2. In particular,

$$A_{vo} = \frac{\beta^2 R_L}{R_S + r_b + r_\pi + (\beta+1)(r_e + R_{EE})} = 2550\ \text{V/V}$$

2. The model pertinent to computing the functions, $Q_S(Z_S, Z_L)$, and $Q_R(Z_S, Z_L)$, is offered in Figure 5.17. Note that the test current source, I_x, which replaces the critical conductance element, G_F, and the resultant test response voltage, V_x, are in disassociated reference polarity. As in Example 5.2, let

$$R_X \triangleq r_e + \frac{R_S + r_b + r_\pi}{\beta+1} = 22.17$$

and

$$\alpha \triangleq \frac{\beta}{\beta+1} = 0.989$$

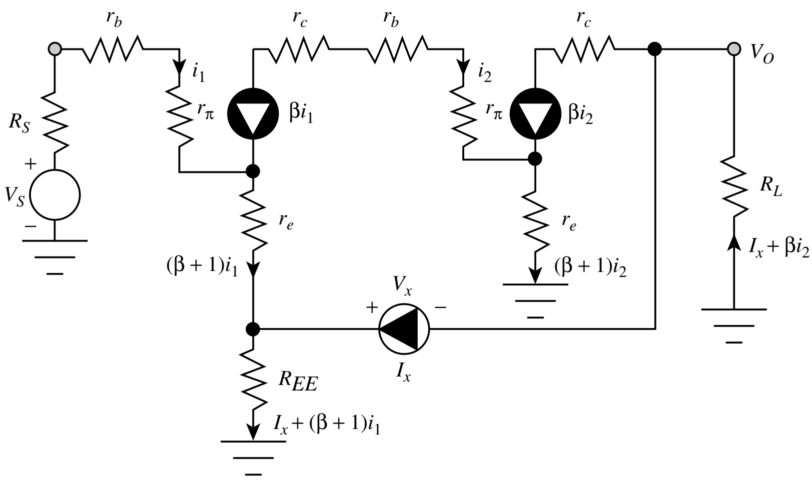

FIGURE 5.17 Circuit used to compute the return ratio and the null return ratio with respect to the conductance, G_F, in the series-shunt feedback amplifier of Figure 5.13(a).

Then, with $V_S = 0$, and writing Q_S (Z_S, Z_L) as Q_S (R_S, R_L) because of the lack of energy storage elements in the circuit undergoing study,

$$Q_S(R_S, R_L) = \left.\frac{V_x}{I_x}\right|_{V_S=0} = R_{EE} \| R_X + \left(1 + \frac{\alpha\beta R_{EE}}{R_{EE} + R_X}\right) R'_L = 258.5\ \text{k}\Omega$$

On the other hand,

$$Q_R(R_S, R_L) = \left.\frac{V_x}{I_x}\right|_{V_O=0} = \left(1 + \frac{1}{\alpha\beta}\right) R_{EE} = 101.1\ \Omega$$

3. Substituting the preceding results into (5.49), and recalling that $G_F = 1/R_F$, the voltage gain of the series-shunt feedback amplifier is found to be

$$A_v = \frac{V_O}{V_S} = A_{vo}\left[\frac{1 + \dfrac{Q_R(R_S, R_L)}{R_F}}{1 + \dfrac{Q_S(R_S, R_L)}{R_F}}\right] = 15.7\ \text{V/V}$$

which is the gain result deduced previously.

Circuit Branch Impedance

In the circuit of Figure 5.18(a), a branch impedance, Z_k, is selected as a critical parameter for analysis; that is $P = Z_k$ in (5.49). The zero parameter gain, $H(0, Z_S, Z_L)$, is evaluated by replacing Z_k with a short circuit, as suggested in Figure 5.18(b).

The volt–ampere characteristic equation of the critical impedance branch is $V_k = Z_k I_k$, where, of course, the branch voltage, V_k, and the branch current, I_k, are in associated reference polarity. Because the dependent variable in this volt–ampere expression is a branch voltage, the return and null return ratios are calculated by replacing the subject branch impedance by a test voltage source, V_x. As suggested in Figure 5.18(c), the ratio, I_x/V_x, under the condition of nulled independent sources, gives the function Q_S (Z_S, Z_L) in (5.49). On the other hand, and as depicted in Figure 5.18(d), the ratio I_x/V_x, with a nulled

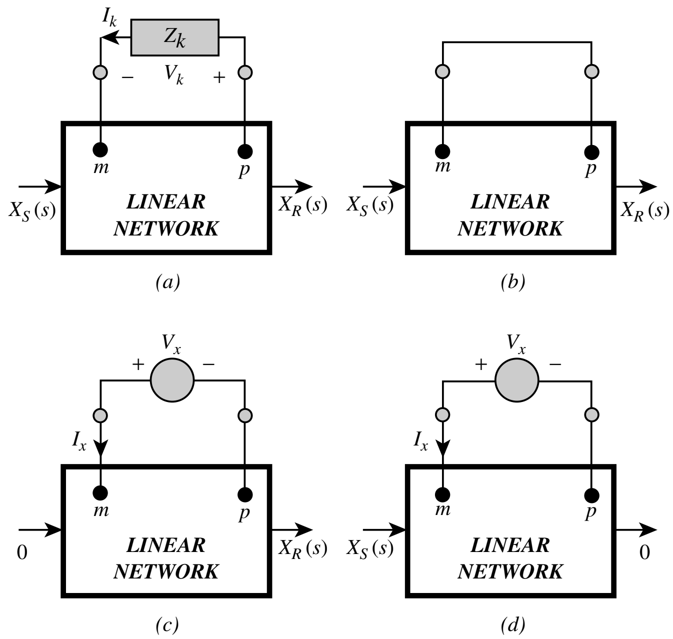

FIGURE 5.18 (a) Linear circuit for which the identified critical parameter is a branch impedance, Z_k. (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain, $H(0, Z_S, Z_L)$. (c) The ratio, I_x/V_x, is the function, $Q_S(Z_S, Z_L)$, in (5.49). (d) The ratio, I_x/V_x, is the function, $Q_R(Z_S, Z_L)$, in (5.49).

response, yields $Q_R(Z_S, Z_L)$. Observe that in the present situation, the functions, $Q_S(Z_S, Z_L)$ and $Q_R(Z_S, Z_L)$ are, respectively, the Thévenin and the null Thévenin admittances facing the branch impedance, Z_k.

Circuit Transimpedance

In the circuit of Figure 5.19(a), a circuit transimpedance, Z_t, is selected as a critical parameter for analysis; that is, $P = Z_t$ in (5.49). The zero parameter gain, $H(0, Z_S, Z_L)$, is evaluated by replacing the current-controlled voltage source (CCVS) by a short circuit, as shown in Figure 5.19(b).

The volt–ampere characteristic equation of the critical transimpedance branch is $V_k = Z_t I_k$, where I_k is the controlling current for the controlled source branch. Because the dependent variable in this volt–ampere expression is a branch voltage, the return and null return ratios are calculated by replacing the CCVS with a test voltage source, V_x. However, as indicated in Figures 5.19(c) and (d), the polarity of V_x mirrors that of the voltage, V_k, developed across the controlled branch. With I_x taken as a current flowing in the controlling branch in a direction opposite to the polarity of the original controlling current, I_k, the ratio, I_x/V_x, under the condition of nulled independent sources, gives the function, $Q_S(Z_S, Z_L)$ in (5.49). On the other hand, and as depicted in Figure 5.19(d), the ratio, I_x/V_x, with a nulled response, yields $Q_R(Z_S, Z_L)$.

Circuit Transadmittance

In the network of Figure 5.20(a), a circuit transadmittance, Y_t, is selected as the critical parameter. The zero parameter gain, $H(0, Z_S, Z_L)$, is evaluated by replacing the voltage-controlled current source (VCCS) with an open circuit, as shown in Figure 5.20(b).

The volt–ampere characteristic question of the critical transadmittance branch is $I_k = Y_t V_k$, where V_k is the controlling voltage for the VCCS. Because the dependent variable in this volt-ampere expression is a branch current, the return and null return ratios are calculated by replacing the VCCS with a test current source, I_x, where, as indicated in Figures 5.20(c) and (d), the polarity of I_x mirrors that of the current, I_k, flowing through the controlled branch. With V_x taken as a voltage developed across the

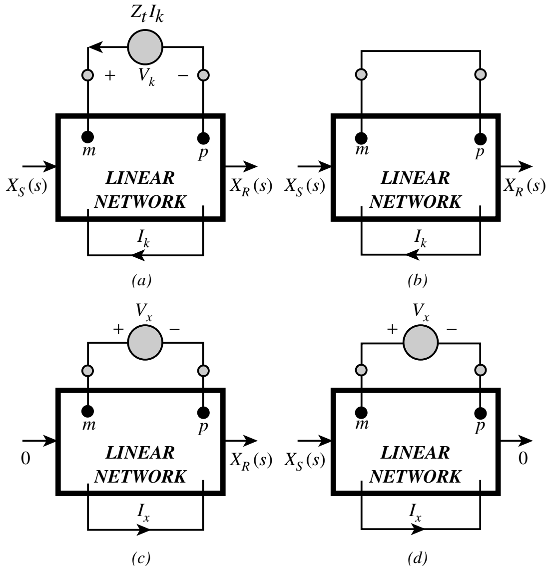

FIGURE 5.19 (a) Linear circuit for which the identified critical parameter is a circuit transimpedance, Z_t. (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain, $H(0, Z_S, Z_L)$. (c) The ratio, I_x/V_x, is the function, $Q_S(Z_S, Z_L)$, in (5.49). (d) The ratio, I_x/V_x, is the function, $Q_R(Z_S, Z_L)$, in (5.49).

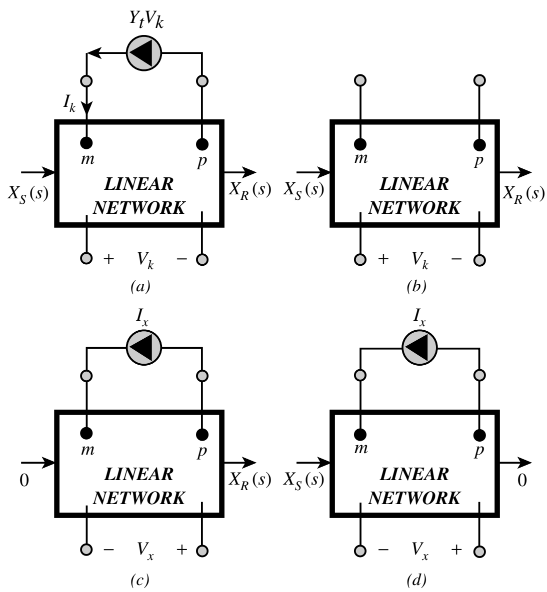

FIGURE 5.20 (a) Linear circuit for which the identified critical parameter is a circuit transadmittance, Y_t, (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain, $H(0, Z_S, Z_L)$. (c) The ratio, V_x/I_x, is the function $Q_S(Z_S, Z_L)$, in (5.49). (d) The ratio, V_x/I_x, is the function $Q_R(Z_S, Z_L)$, in (5.49).

controlling branch in a direction opposite to the polarity of the original controlling voltage, V_k, the ratio, V_x/I_x, under the condition of nulled independent sources, gives the function, $Q_S(Z_S, Z_L)$ in (5.43). On the other hand, and as offered in Figure 5.20(d), the ratio, V_x/I_x, under the condition of a nulled response, yields $Q_R(Z_S, Z_L)$.

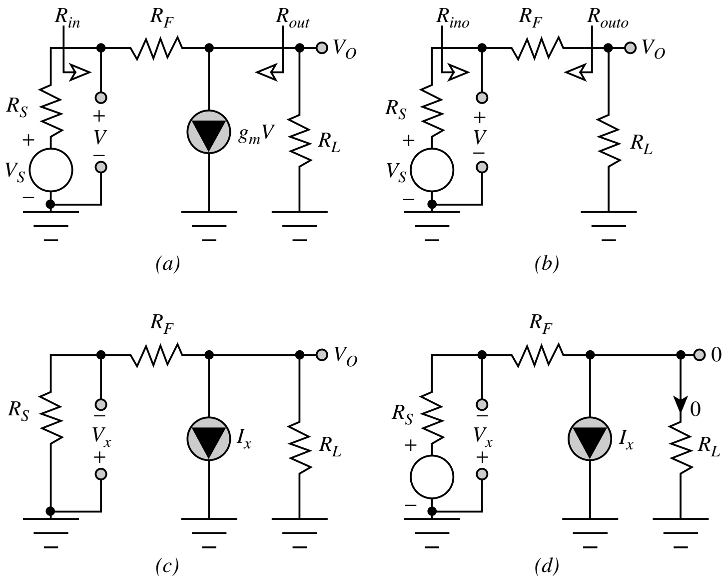

FIGURE 5.21 (a) The low-frequency, small-signal model of a voltage feedback amplifier. (b) The circuit used to evaluate the zero parameter ($g_m = 0$) gain. (c) The circuit used to evaluate the return ratio with respect to g_m. (d) The circuit used to computer the null return ratio with respect to g_m.

Example 5.4. The circuit in Figure 5.21(a) is a low-frequency, small-signal model of a voltage feedback amplifier. With the transconductance, g_m, selected as the reference parameter of interest, derive a general expression for the voltage gain, $A_v = V_O/V_S$. Approximate the final result for the special case of very large g_m.

Solution.

1. The zero parameter voltage gain, A_{vo}, derives from an analysis of the circuit structure given in Figure 5.21(b). The diagram differs from Figure 5.21(a) in that the current conducted by the controlled source branch has been nulled by open circuiting said branch. By inspection of the subject model,

$$A_{vo} = \frac{R_L}{R_L + R_F + R_S}$$

2. The diagram given in Figure 5.21(c) is appropriate to the computation of the return ratio, $T_S(g_m, Z_S, Z_L) = g_m Q_S(R_S, R_L)$ with respect to the critical transconductance, g_m. A comparison of the model at hand with the diagram in Figure 5.21(a) confirms that the controlled source, g_mV, is replaced by an independent current source, I_x, which flows in a direction identical to that of the controlled source it supplants. The ratio, V_x/I_x, is to be computed, where V_x is developed, antiphase to V, across the branch that supports the original controlling voltage for the VCCS. A straightforward analysis produces

$$Q_S(R_S, R_L) = \frac{V_x}{I_x}\left(\frac{R_L}{R_L + R_F + R_S}\right)R_S \equiv A_{vo}R_S$$

3. The null return ratio, $T_R(g_m, Z_S, Z_L) = g_m Q_R(R_S, R_L)$, with respect to g_m is obtained from an analysis of the circuit in Figure 5.21(d). Observe a nulled output voltage, with zero current flow through the load resistance, R_L. Observe further that the signal source voltage is nonzero. The specific value of this source voltage is not crucial, and is therefore not delineated. An analysis reveals

$$Q_R(R_S, R_L) = \frac{V_x}{I_x} = -R_F$$

4. Using (5.49), the voltage gain of the circuit undergoing study is found to be

$$A_v = \frac{V_O}{V_S} = A_{vo}\left[\frac{1 + g_m(-R_F)}{1 + g_m R_S A_{vo}}\right]$$

which, for large g_m, reduces to

$$A_v \approx -\frac{R_F}{R_S}$$

Gain of a Current-Controlled Current Source

For the network in Figure 5.22(a), the reference parameter is α_k, the gain associated with a CCCS. The zero parameter gain, $H(0, Z_S, Z_L)$, is evaluated by replacing this CCCS with an open circuit, as depicted in Figure 5.22(b).

The volt–ampere characteristic equation of the branch in which the reference parameter is embedded is $I_k = \alpha_k I_j$, where I_j is the controlling current for the CCCS. Because the dependent variable in this volt-ampere characteristic is a branch current, the return and null return ratios are calculated by replacing the CCCS with a test current source, I_x. As indicated in Figures 5.22(c) and (d), the polarity for I_x mirrors that of the current, I_k, flowing through the controlled branch. Let I_y be the resultant current conducted by the controlling branch, and let this current flow a direction opposite to the polarity of the original controlling current. Then, the current ratio, I_y/I_x, computed under the condition of nulled independent sources, is the function, $Q_S(Z_S, Z_L)$ in (5.49). Similarly, and as suggested in Figure 5.22(d), the ratio, I_y/I_x, under the condition of a nulled response, yields $Q_R(Z_S, Z_L)$.

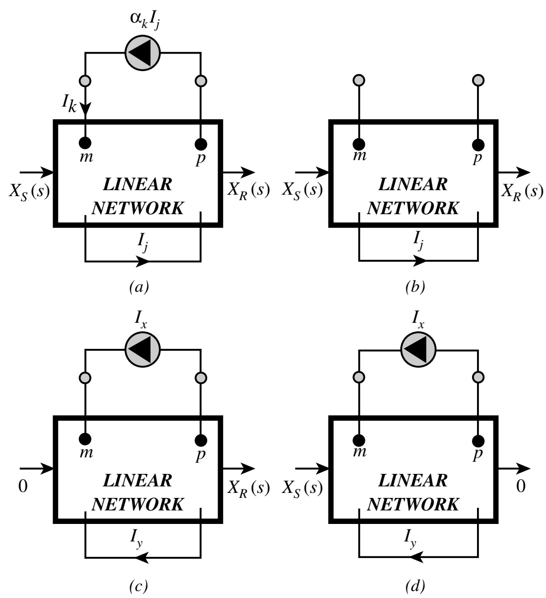

FIGURE 5.22 (a) Linear circuit for which the identified critical parameter is the current gain α_k associated with a CCCS. (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain, $H(0, Z_S, Z_L)$. (c) The ratio, I_y/I_x, is the function, $Q_S(Z_S, Z_L)$, in (5.49). (d) The ratio, I_y/I_x, is the function, $Q_R(Z_S, Z_L)$, in (5.49).

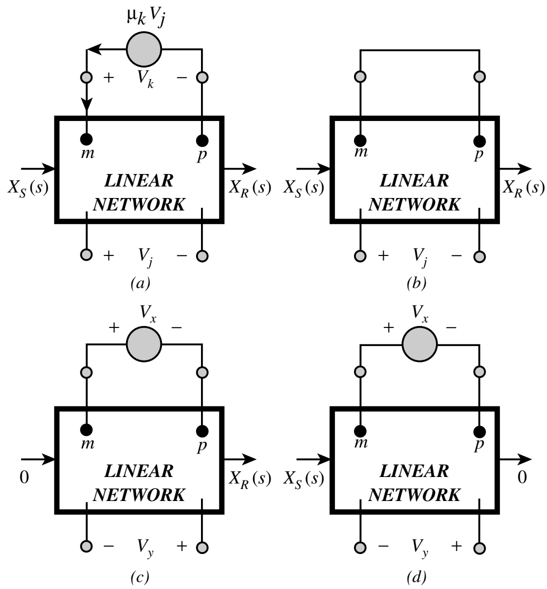

FIGURE 5.23 (a) Linear circuit for which the identified critical parameter is the voltage gain, μ_{k1} associated with a VCVS. (b) The ratio, $X_R(s)/X_S(s)$, is the zero parameter gain, $H(0, Z_S, Z_L)$. (c) The ratio, V_y/V_x, is the function Q_S (Z_S, Z_L), in (5.49). (d) The ratio, V_y/V_x, is the function, $Q_R(Z_S, Z_L)$, in (5.49).

Gain of a Voltage-Controlled Voltage Source

In the network of Figure 5.23(a), the selected reference parameter is μ_k, the gain corresponding to a VCVS. The zero parameter gain, $H(0, Z_S, Z_L)$, is evaluated by replacing this VCVS with a short circuit, as per Figure 5.23(b).

The volt–ampere characteristic equation of the dependent generator branch is $V_k = \mu_k V_j$, where V_j is the controlling voltage for the VCVS. Because the dependent variable in this volt–ampere expression is a branch voltage, the return and null return ratios are calculated by replacing the VCVS with a test voltage source, V_x, where as indicated in Figures 5.23(c) and (d), the polarity of V_x is identical to that of the voltage, V_k, developed across the controlled branch. Let V_y be the resultant voltage established across the controlling branch, and let the polarity of this voltage be in a direction opposite to that of the original controlling voltage. Then, the voltage ratio, V_y/V_x, computed under the condition of nulled independent sources, is the function, $Q_S(Z_S, Z_L)$, in (5.49). As suggested in Figure 5.23(d), the voltage ratio, V_y/V_x, under the condition of a nulled response, yields $Q_R(Z_S, Z_L)$.

Evaluation of Driving Point Impedances

Having formulated generalized techniques for computing the return ratio and the null return ratio with respect to any of the six possible types of critical circuit parameters, the application of (5.49) is established as a powerful and computationally expedient vehicle for evaluating any transfer function of any linear network. The only restriction limiting the utility of (5.49) is that parameter P must be selected in such a way as to ensure that the zero parameter transfer function is finite and nonzero.

Equation (5.49) is commonly used to evaluate the voltage gain, current gain, transimpedance gain, or transadmittance gain of feedback and other types of complex circuitry. However, the expression is equally well suited to determining the driving point input impedance seen by the source impedance, as well as the driving point output impedance seen by the terminating load impedance. In fact, once the return ratios relevant to the gain of interest are found, these I-O impedances can be determined with minimal additional analysis.

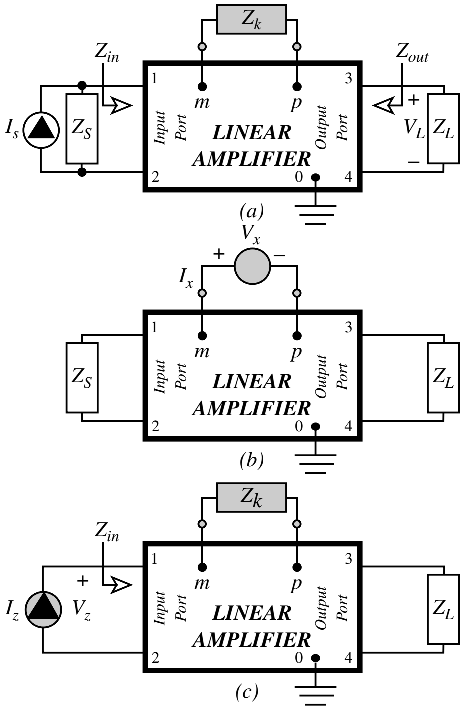

FIGURE 5.24 (a) A liner amplifier for which the input impedance is to be determined. (b) The circuit used for calculating the return ratio with respect to Z_k. (c) The circuit used for calculating the driving point input impedance.

Without loss of generality, the foregoing contention is explicitly demonstrated in conjunction with a transimpedance amplifier whose reference parameter is selected to be a branch impedance, Z_k. To this end, consider the circuit abstracted in Figure 5.24, for which the driving point input impedance, Z_{in}, is to be determined. The input excitation is a current, I_S, and in response to this input, a signal voltage, V_L, is developed across the load impedance, Z_L. Using (5.49), the I-O transimpedance, $Z_T\,(Z_k, Z_S, Z_L)$, is

$$Z_T\left(Z_K, Z_S, Z_L\right) = \frac{V_L(s)}{I_S(s)} = Z_T\left(0, Z_S, Z_L\right)\left[\frac{1 + Z_k Q_R\left(Z_S, Z_L\right)}{1 + Z_k Q_S\left(Z_s, Z_L\right)}\right] \tag{5.55}$$

where $Z_T\,(0, Z_S, Z_L)$ is the circuit transimpedance for $Z_k = 0$, $Z_k\,Q_R\,(Z_S, Z_L)$ is the null return ratio with respect to Z_k, and $Z_K\,Q_S\,(Z_S, Z_L)$ is the return ratio with respect to Z_k. For future reference, the circuit appropriate to the calculation of the function, $Q_S\,(Z_S, Z_L)$, is drawn in Figure 5.24(b).

The input impedance derives from an analytical consideration of the cell depicted in Figure 5.24(c), in which the Norton representation of the signal source has been supplanted by a test current source of value I_z. Note that the load impedance remains as the terminating element for the output port. The transfer relationship of interest is the ratio, V_z/I_z, which is the desired driving point input impedance, Z_{in}. Taking care to choose Z_k, the reference parameter for the gain enumeration, as the reference parameter for the input impedance determination, (5.49) gives

$$Z_{\text{in}} = \frac{V_z}{I_z} = Z_{\text{ino}}\left[\frac{1 + Z_k Q_{RR}\left(Z_S, Z_L\right)}{1 + Z_k Q_{SS}\left(Z_S, Z_L\right)}\right] \tag{5.56}$$

In this expression, Z_{ino} generally derives straightforwardly because it is the $Z_k = 0$ value of Z_{in}; that is, Z_{in} is evaluated for the special case of a nulled reference parameter. Such a null in the present situation is equivalent to short-circuiting Z_k, as indicated in Figure 5.25(a). The function, $Q_{SS}\,(Z_S, Z_L)$, is the delineated

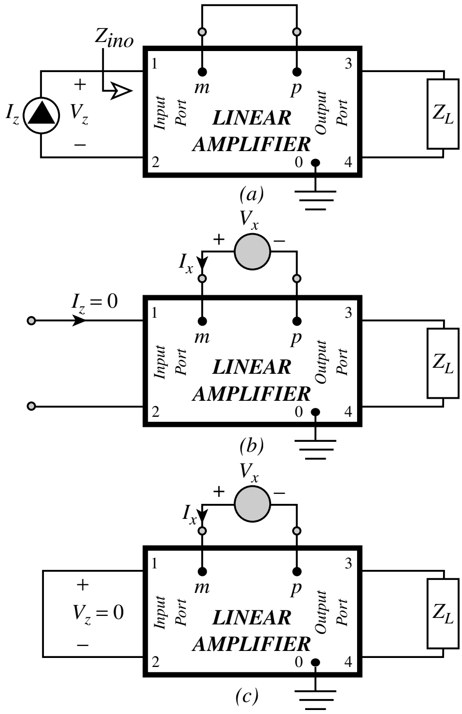

FIGURE 5.25 (a) The circuit used to evaluate the zero parameter driving point input impedance. (b) The computation, relative to input impedance, of the return ratio with respect to Z_k. (c) The computation, relative to input impedance, of the null return ratio with respect to Z_k.

I_x/V_x ratio, for the case of a source excitation (I_z in the present case) set to zero. The pertinent circuit diagram is the structure in Figure 5.25(b). This last circuit differs from the circuit, shown in Figure 5.24(b), exploited to find Q_S (Z_S, Z_L) in the gain relationship of (5.50) in only one way: Z_S has been removed, and thus, effectively, Z_S has been set to an infinitely large value. It follows that

$$Q_{SS}(Z_S, Z_L) \equiv Q_S(\infty, Z_L) \tag{5.57}$$

In other words, a circuit analysis aimed toward determining Q_{SS} (Z_S, Z_L) is unnecessary. Instead, Q_{SS} (Z_S, Z_L) is found by evaluating $Q_S(Z_S, Z_L)$, which is already known from the gain analysis, *at* $Z_S = \infty$.

To evaluate Q_{RR} (Z_S, Z_L), the foregoing I_x/V_x ratio is calculated for the case of zero response. In the present situation the response is the voltage, V_z, and accordingly, the appropriate circuit is depicted in Figure 5.25(c). However, a comparison of the circuit at hand with the structure in Figure 5.24, which is exploited to evaluate the return ratio in the gain equation, indicates that it differs only in that Z_S is now constrained to zero to ensure $V_z = 0$. It is therefore apparent that in (5.56)

$$Q_{RR}(Z_S, Z_L) \equiv Q_S(0, Z_L) \tag{5.58}$$

Equation (5.56) is now expressible as

$$Z_{\text{in}} = \frac{V_z}{I_z} = Z_{\text{ino}}\left[\frac{1 + Z_k Q_S(0, Z_L)}{1 + Z_k Q_S(\infty, Z_L)}\right] \tag{5.59}$$

which is occasionally referred to as Blackman's formula [14].

Analogous considerations at the output port in the circuit of Figure 5.24(a) dictate a driving point output impedance, Z_{out}, of

$$Z_{out} = Z_{outo}\left[\frac{1+Z_kQ_S(Z_S,0)}{1+Z_kQ_S(Z_S,\infty)}\right] \tag{5.60}$$

where, similar to Z_{ino}, Z_{outo}, the $Z_k = 0$ value of Z_{out}, must be finite and nonzero. Although the preceding two relationships were derived for the case in which the selected reference parameter is a branch impedance, both expressions are applicable for any reference parameter, P. In general,

$$Z_{in} = Z_{ino}\left[\frac{1+PQ_s(0,Z_L)}{1+PQ_S(\infty,Z_L)}\right] \tag{5.61a}$$

$$Z_{out} = Z_{outo}\left[\frac{1+PQ_s(Z_S,0)}{1+PQ_S(Z_S,\infty)}\right] \tag{5.61b}$$

Example 5.5. Use the pertinent results of Example 5.4 to derive expression for the driving point input resistance, R_{in}, and the driving point output resistance, R_{out}, of the feedback amplifier in Figure 5.21(a).

Solution.

1. With g_m set to zero, an inspection of the circuit diagram in Figure 5.21(b) delivers

$$R_{ino} = R_F + R_L$$

$$R_{outo} = R_F + R_S$$

2. From the second step in the solution to Example 5.4, the function, $Q_S(R_S, R_L)$, to which the return ratio, $T_S(g_m, R_S, R_L)$ is directly proportional, was found to be

$$Q_S(R_S,R_L) = \left(\frac{R_L}{R_L+R_F+R_S}\right)R_S$$

It follows that

$$Q_S(0,R_L) = 0$$

$$Q_S(\infty,R_L) = R_L$$

Moreover,

$$Q_S(R_S,0) = 0$$

$$Q_S(R_S,\infty) = R_S$$

3. Equations (5.61a) and (b) resultantly yield

$$R_{in} = R_{ino}\left[\frac{1+g_mQ_S(0,Z_L)}{1+g_mQ_S(\infty,Z_L)}\right] = \frac{R_R+R_L}{1+g_mR_L}$$

for the driving point input resistance and

$$R_{\text{out}} = R_{\text{outo}}\left[\frac{1+g_m Q_S(R_S,0)}{1+g_m Q_S(R_S,\infty)}\right] = \frac{R_F + R_S}{1+g_m R_S}$$

for the driving point output resistance.

Sensitivity Analysis

Yet another advantage of the Kron–Bode formula is its amenability to evaluating the impact exerted on a circuit transfer relationship by potentially large fluctuations in the reference parameter P. This convenience stems from the fact that parameter P is isolated in (5.49); that *is*, $H\,(0, Z_S, Z_L)$, $Q_R\,(Z_S, Z_L)$, and (Z_S, Z_L) are each independent of P. A quantification of this impact is achieved by exploiting the *sensitivity function*,

$$S_P^H \triangleq \frac{\Delta H/H}{\Delta P/P} \tag{5.62}$$

which compares the per unit change in transfer function, $\Delta H/H$, resulting from a specified per unit change $\Delta P/P$ in a critical parameter. In particular, the notation in this definition is such that if H designates the transfer characteristic, $H\,(P_0, Z_S, Z_L)$, at the nominal parameter setting, $P = P_0$, $(H + \Delta H)$ signifies the perturbed characteristic, $H(P_0 + \Delta P, Z_S, Z_L)$ where P_0 is altered by an amount ΔP_0. Using (5.49) and dropping the functional notation in (5.53a) and (5.53b), it can be demonstrated that

$$S_P^H \triangleq \frac{F_R - F_S}{F_R\left[F_S + (F_S - 1)\left(\dfrac{\Delta P}{P_0}\right)\right]} \tag{5.63}$$

where F_S and F_R are understood to be evaluated at the nominal parameter setting, $P = P_0$. It should be emphasized that unlike a more traditional sensitivity analysis, such as that predicated on the adjoint network [15], (5.63) is easy to use manually and does not rely on an *a priori* assumption of small parametric changes.

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