

# 6

## Tableau and Modified Nodal Formulations

---

6.1	Introduction .....	6-1
6.2	Nodal and Mesh Formulations .....	6-1
6.3	Graphs and Tableau Formulation .....	6-5
6.4	Modified Nodal Formulation .....	6-8
6.5	Nonlinear Elements.....	6-14
6.6	Nodal Analysis of Active Networks.....	6-17

Jiri Vlach

*University of Waterloo, Canada*

### 6.1 Introduction

---

Network analysis is based on formulation of the relevant equations and on their solutions. Various approaches are possible. If we wish to get as much theoretical information as possible, we may resort to hand analysis and keep the elements as variables (literal parameters). In such cases, it is an absolute necessity to use a method that leads to the smallest possible number of equations. If we plan to use a computer, then we can accept methods which lead to larger systems, but the methods must be relatively easy to program. The purpose of this chapter is to give an overview of the various possibilities and point out advantages and disadvantages. Many more details are available in [1, 2].

Section 6.2 is a summary of the nodal and mesh formulations. We review them because they are the best ones for analysis of small networks. Tableau formulation, given in Section 6.3, is very general, but requires special solution routines, probably not available to the reader. Section 6.4 describes the best method for computerized solutions; it is used in many commercial simulators. If nonlinear elements are involved, then iterative solution methods must be used; an introduction on how to deal with nonlinear elements is given in Section 6.5. Finally, Section 6.6 presents a method that is suitable for hand solutions of active networks and which automatically leads to the smallest system of equations.

### 6.2 Nodal and Mesh Formulations

---

Classical methods use two types of network equations formulation: the nodal and the mesh. The first one is based on Kirchhoff's current law (KCL): the sum of currents flowing away from a node is equal to zero. The mesh method is based on Kirchhoff's voltage law (KVL): the sum of voltages around any loop is equal to zero.

For simple problems, both methods are about equivalent, but nodal formulation is more general. The mesh formulation is suitable only for planar networks: It must be possible to draw the network without any element crossing over any other element. Many practical networks are planar, but it is not always easy to see that it is actually the case.

We first introduce some definitions. A positive current flows from a terminal with a higher potential to a terminal with a lower potential. This is sketched on the two-terminal element in [Figure 6.1](#). If we

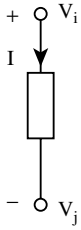


FIGURE 6.1 Definition of positive current direction with respect to the voltage across the element.

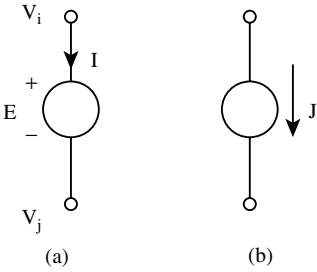


FIGURE 6.2 (a) Direction of positive current through an independent voltage source. (b) Direction of the current through an independent current source.

use this definition, then the product of the current and of the voltage across the element,  $V_i - V_j$ , expresses the power consumed by the element. If the current flows in opposite direction, then the element is delivering power.

We will use the previous definition of a positive current for *all* elements of the network, irrespective of what their role eventually may be. Thus, a positive current through an independent voltage source will flow from the more positive terminal to the less positive terminal, as sketched in Figure 6.2(a). The current flowing through an independent current source is indicated on its symbol, Figure 6.2(b), but the voltage across it is not defined; it depends on the network.

The nodal formulation uses the principle that the sum of currents at any node must be equal to zero at any instant of time. To apply this rule in an efficient way, we realize that before we solve the equations, we do not know which way the currents will actually flow. All we know is that *if* a node is more positive than all the other nodes, then all currents must flow *away* from this node.

In nodal formulation, the unknowns are nodal voltages and the equations express the sum of currents flowing away from the node. To write the equations we use element admittances: in Laplace transform  $Y_C = sC$  for a capacitor,  $Y_L = 1/sL$  for an inductor, and  $Y_G = G = 1/R$  for a resistor. It is advantageous to use  $G$ , because we thus avoid fractions in the equations.

We demonstrate how to set up the equations by considering the network in Figure 6.3. The nodal voltages are denoted  $V_1$  and  $V_2$  and ground (the lower line) is considered to be at zero potential. We do not know which of these nodes is more positive, but we can *assume* that any node we consider at a given moment is the most positive one. This assumption has the consequence that all currents must flow *away*

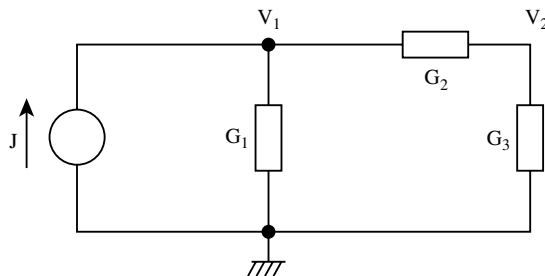


FIGURE 6.3 Example for nodal formulation.

from this node. For the given network, the current through  $G_1$  will flow down and its value will be  $I_{G_1} = G_1 V_1$ , current through  $G_2$  will flow from left to right and will be  $I_{G_2} = G_2 (V_1 - V_2)$ . Current from the independent current source flows *into* the node and thus must be subtracted. Together, the sum of the currents at node 1 will be zero:

$$G_1 V_1 + G_2 (V_1 - V_2) - J = 0$$

Moving to the second node, we still do not know which node is more positive, but we can still make the assumption that *now* it is *this* node that is the most positive one. If we make such an assumption, then all currents must flow away from this node: The current through  $G_3$  will be  $I_{G_3} = G_3 V_2$  and will flow down, the current through  $G_2$  will flow from right to left and will be  $I_{G_2} = G_2 (V_2 - V_1)$ . In this expression, the first voltage within the parentheses must be the assumed higher potential. This is expressed by the equation

$$G_3 V_2 + G_2 (V_2 - V_1) = 0$$

It is advantageous to put the equations into matrix form, with the known independent source transferred to the right side:

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix}$$

The preceding steps were simple because we selected elements that can be handled by this formulation. Unfortunately, many practical elements are not expressed in terms of currents. For instance, a voltage source connected between nodes  $i$  and  $j$ , with its positive reference on node  $i$ , is described by the equation

$$V_i - V_j = E$$

A positive current does flow through such an element from  $i$  to  $j$ , but is not available in its defining equation. In fact, all voltage sources, independent or dependent, will create this problem. Another element which cannot be handled directly is a short circuit. It is described by the equation

$$V_i - V_j = 0$$

and current is not a part of its definition.

We can always use transformations by applying various theorems such as the Thévenin and Norton transformations or source splitting, and eventually arrive at a network in which all elements have voltage as the independent variable. Such transformations are practical for hand analysis, but are not advantageous for computerized solutions. This is the reason why other formulations have been invented.

Consider next the mesh equations where we use the KVL and impedances of the elements:  $Z_L = sL$  for the inductor,  $Z_C = 1/sC$  for the capacitor, and  $R$  for the resistor. In this formulation, we sum the voltages across the elements in a given closed loop. Because this method is suitable for planar networks only, we usually use the concept of circulating mesh currents, indicated on the network in [Figure 6.4](#). The currents  $I_1$  and  $I_2$  create voltage drops across the resistors. When considering the first mesh, we take the current  $I_1$  as a positive one. The voltage across  $R_1$  is  $V_{R_1} = R_1 I_1$ . The voltage across  $R_2$  is  $V_{R_2} = R_2 (I_1 - I_2)$  and the voltage source contributes a value  $E$  to the equation. According to our earlier definition, a positive current flows from plus to minus, but  $I_1$  actually goes in the opposite direction through the voltage source. Thus, the voltage across  $E$  must be taken with a negative sign and the sum of voltages around the first mesh is

$$R_1 I_1 + R_2 (I_1 - I_2) - E = 0$$

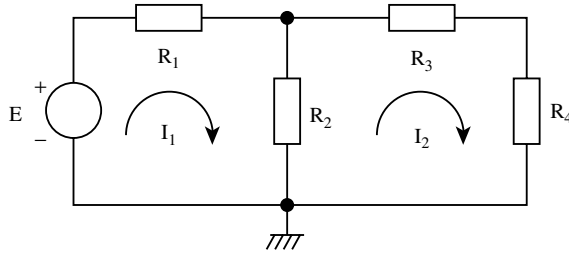


FIGURE 6.4 Example for mesh formulation.

When we move to the second mesh, we consider the current  $I_2$  as positive and the sum of voltage drops around the second mesh is

$$R_2(I_2 - I_1) + R_3I_2 + R_4I_2 = 0$$

The equations can be collected into one matrix equation

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

Each of these fundamental formulations has its problems.

In nodal formulation, we can deal directly with the following elements:

- Current source,  $J$
- Conductance,  $G = 1/R$
- Capacitor admittance,  $sC$
- Voltage controlled current source,  $VC$
- Inductor admittance,  $1/sL$

In mesh formulation, we can deal directly with the elements:

- Voltage source,  $E$
- Resistor,  $R$
- Inductor impedance,  $sL$
- Current controlled voltage source,  $CV$
- Capacitor impedance,  $1/sC$ .

All other elements create problems and must be dealt with by the Thévenin and Norton theorems and/or source splitting.

As an example, we take the network in Figure 6.5(a). It is directly suitable for mesh formulation, but we demonstrate both. For simplicity, all resistors have unit values.

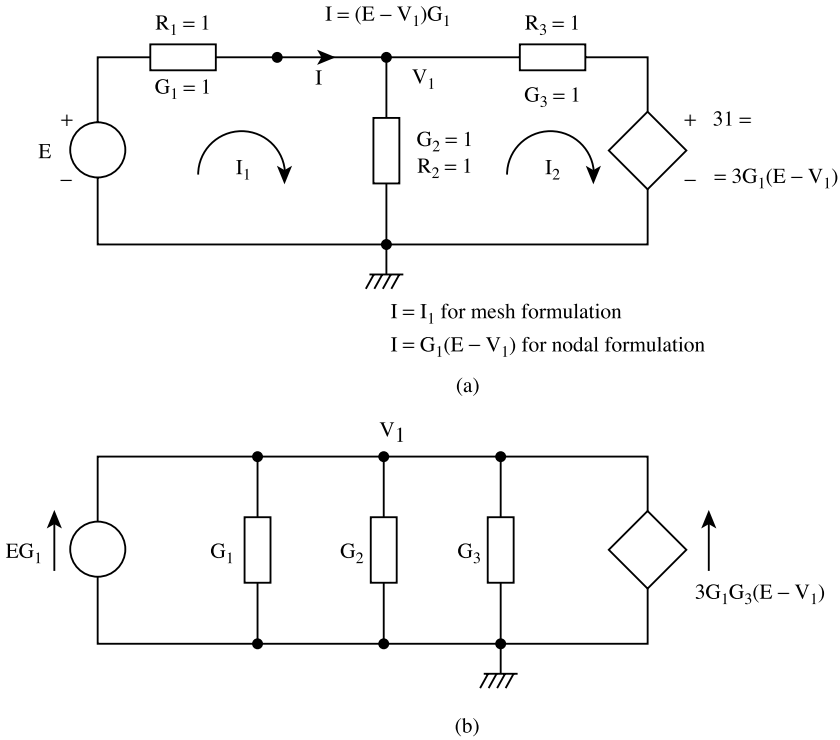
The mesh formulation, with the indicated circulating currents  $I_1$  and  $I_2$ , leads to the equations

$$\begin{aligned} (R_1 + R_2)I_1 - R_2I_2 &= E \\ -R_2I_1 + (R_2 + R_3)I_2 + 3I_1 &= 0 \end{aligned}$$

Inserting numerical values

$$\begin{aligned} 2I_1 - I_2 &= E \\ 2I_1 + 2I_2 &= 0 \end{aligned}$$

The solution is  $I_1 = E/3$ ,  $I_2 = -E/3$  and  $V_1 = R_2(I_1 - I_2) = 2E/3$ .



**FIGURE 6.5** (a) Example of a network suitable for mesh formulation. (b) Modification of the network in Figure 6.5(a) to be suitable for nodal formulation.

To use nodal formulation, we must apply several transformation steps. First, we must express the controlling currents as  $I = G_1(E - V_1) = E - V_1$  and replace  $I$  in the definition of the current controlled voltage source. This has been done in the figure. Afterward, applying Thévenin–Norton transformation we change the voltage sources, in series with resistors, into current sources, in parallel with the same resistors. We get the network in Figure 6.5(b). It has only one node for which the balance of current is

$$(G_1 + G_2 + G_3)V_1 - EG_1 - 3G_1G_3(E - V_1) = 0$$

Inserting numerical values and solving, we get the same  $V_1$  as previously.

If we have a mixture of elements, such transformations will always be lengthy, will require redrawings, and can lead to errors. To reduce the chance of such errors to a minimum, in computer applications we need formulations that avoid transformations and use descriptions of the elements as they are given. This is done in both the tableau and nodal formulations, the subjects of the following sections.

Although the nodal and mesh formulations are not always easy to apply, we must stress that they are the best ones for hand solutions. They may require several steps of transformations and redrawings, but ultimately they lead to the smallest possible systems of equations.

### 6.3 Graphs and Tableau Formulation

Tableau is the most general formulation because the solution simultaneously provides the voltages across all elements, the currents through all elements, and all nodal voltages. The difficulty is that tableau leads to such large systems of equations that complicated sparse matrix solvers are an absolute necessity. Most readers will not have access to such routines, therefore, we will explain its properties only to the extent necessary for understanding. We advise the reader not to use it.

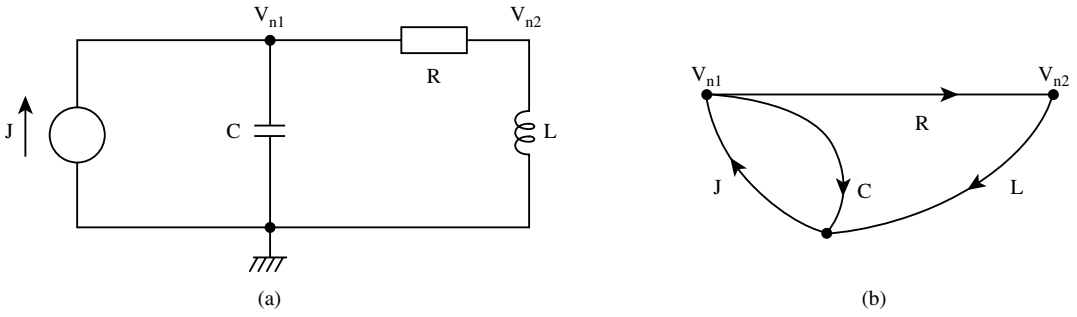


FIGURE 6.6 (a) A simple network; (b) its graph.

Tableau formulation needs for its construction the concept of graphs and the concept of the incidence matrix. Consider the network in Figure 6.6(a). A graph of the network replaces each element by a line. We will use oriented graphs, with arrows, because they can be identified with the flow of currents. In all passive elements, the current can flow in any direction and the orientation of the graph is entirely our choice. We do not have such freedom when we consider sources. The direction of the current through the current source is given by the arrow marked at its symbol and we use the same direction in the graph. For the voltage source, the direction of the graph will be from plus to minus, in agreement with our previous explanations. Each node is marked by the node voltage and the line representing the element is given the name of the element. Following these rules, we have constructed the graph in Figure 6.6(b). Because the directions of the arrows are the assumed directions of currents, we can write KCL for the two nodes: positive direction is away from the node, negative is into the node. The sums of currents for the nodes are

$$\begin{aligned}
 -I_J + I_C + I_R &= 0 \\
 -I_R + I_L &= 0
 \end{aligned}$$

This can also be summarized in one matrix equation

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_J \\ I_C \\ I_R \\ I_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\mathbf{AI} = 0 \tag{6.1}$$

The matrix  $A$  is called the *incidence* matrix. It has as many rows as there are ungrounded nodes, and as many columns as the number of elements. Note that  $+1$  in any given row indicates that we expect the current to flow away from the node,  $-1$  means the opposite.

Still more information can be extracted from this matrix. Denote the nodal voltages by subscripts  $n$  in  $V_{n1}$  and  $V_{n2}$ , as done in Figure 6.6. The voltages across the elements will have as subscripts the names of the elements. We can write the following set of equations which couple the voltages across the elements with the nodal voltages:

$$\begin{aligned}
 V_J &= -V_{n1} \\
 V_C &= V_{n1}
 \end{aligned}$$

$$V_R = V_{n1} - V_{n2}$$

$$V_L = V_{n2}$$

In matrix form, this is equivalent to

$$\begin{bmatrix} V_J \\ V_C \\ V_R \\ V_L \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_{n1} \\ V_{n2} \end{bmatrix}$$

The matrix is the transpose of the incidence matrix and we can generalize

$$\mathbf{V}_{el} - \mathbf{A}^T \mathbf{V}_n = 0 \tag{6.2}$$

Complete formulation needs expressions that couple the element currents and the element voltages. Writing them in the same sequence as for the graph, and using Laplace transformation, we have

$$I_J = J$$

$$I_C = sCV_C$$

$$V_R = RI_R$$

$$V_L = sLI_L$$

For matrix notation, we need an expression that, with a proper choice of entries, will cover all possible elements. Such an expression is

$$YV_{el} + ZI_{el} = W$$

For instance, if we consider the current source, we set  $Y = 0$ ,  $Z = 1$  and  $W = J$ , which gives the preceding equation. Similar choices can be made for the other elements.

The KCL equation,  $\mathbf{AI} = \mathbf{0}$ , the KVL equation,  $\mathbf{V}_{el} - \mathbf{A}^T \mathbf{V}_n = \mathbf{0}$ , and the previous equation  $\mathbf{YV}_{el} + \mathbf{ZI}_{el} = \mathbf{W}$  are collected in one matrix equation. Any sequence can be used; we have chosen

$$\begin{aligned} \mathbf{V}_{el} - \mathbf{A}^T \mathbf{V}_n &= \mathbf{0} \\ \mathbf{YV}_{el} + \mathbf{ZI}_{el} &= \mathbf{W} \\ \mathbf{AI}_{el} &= \mathbf{0} \end{aligned} \tag{6.3}$$

and in matrix form

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & -\mathbf{A}^T \\ \mathbf{Y} & \mathbf{Z} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{el} \\ \mathbf{I}_{el} \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{W} \\ \mathbf{0} \end{bmatrix} \tag{6.4}$$

Once the incidence matrix is available, writing this matrix equation is actually quite simple. First determine its size: it will be twice the number of elements plus the number of nodes. For our example, it will be 10. The system equation is in [Figure 6.7](#) where all zero entries were omitted to clearly show the

$$\begin{array}{c}
 V_j \quad V_c \quad V_R \quad V_L \quad I_j \quad I_C \quad I_R \quad I_L \quad V_{n1} \quad V_{n2} \\
 \hline
 \begin{bmatrix}
 1 & & & & & & & & 1 & & \\
 & 1 & & & & & & & -1 & & \\
 & & 1 & & & & & & -1 & 1 & \\
 & & & 1 & & & & & & -1 & \\
 \hline
 & & & & 1 & & & & & & \\
 & sC & & & & -1 & & & & & \\
 & & 1 & & & & R & & & & \\
 & & & 1 & & & & sL & & & \\
 & & & & -1 & 1 & 1 & & & & \\
 & & & & & & -1 & 1 & & & 
 \end{bmatrix}
 \begin{bmatrix}
 V_j \\
 V_c \\
 V_R \\
 V_L \\
 I_j \\
 I_C \\
 I_R \\
 I_L \\
 V_{n1} \\
 V_{n2}
 \end{bmatrix}
 =
 \begin{bmatrix}
 \\
 \\
 \\
 \\
 \\
 \\
 \\
 \\
 \\
 \\
 \end{bmatrix}
 J
 \end{array}$$

FIGURE 6.7 Tableau formulation for the network in Figure 6.6.

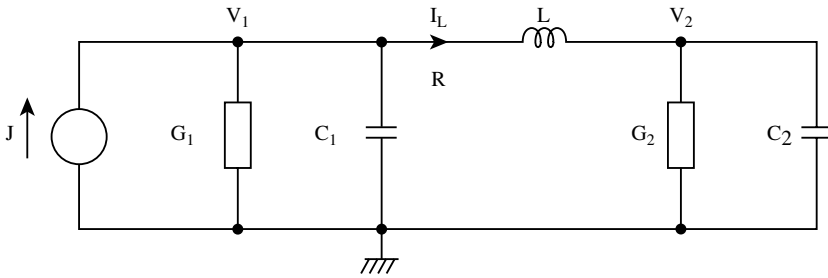


FIGURE 6.8 Example for modified nodal formulation.

structure. In the top partition is a unit matrix and the negative of the transpose of the **A** matrix. In the bottom partition is the incidence matrix, **A**. The middle portion is filled, element by element, using the previous element equation. For better understanding, it is a good idea to write the variables above the matrix, as shown, because each column of the matrix is multiplied by the variable which appears above it.

We have used this simple example to point out the main difficulty of the tableau formulation: the system becomes very large. In nodal formulation, this problem would lead to only two equations. However, the tableau system matrix has many zeros and is said to be sparse. Sparse systems are always solved by special routines which, roughly speaking, do not store the zeros and do not operate on them. Such codes are quite difficult to write, and in tableau we have the additional difficulty that the matrix has a complicated structure. We discussed this formulation more as a warning instead of a recommendation. Unless a suitable sparse matrix solver is already available, this formulation should be avoided.

### 6.4 Modified Nodal Formulation

Modified nodal formulation is an extension of the nodal formulation and is the method of choice for computerized analysis. It is used in most commercial simulators, and we will explain it in considerable detail.

When nodal formulation is taught in schools, inductors are usually taken as admittances,  $Y_L = 1/sL$ . This is fine, as long as we work by hand and derive the network function. For instance, nodal equations for the network in Figure 6.8 would be written in the form



$$\begin{aligned} \left(G_1 + sC_1 + \frac{1}{sL}\right)V_1 - \frac{1}{sL}V_2 &= J \\ -\frac{1}{sL}V_1 + \left(G_2 + sC_2 + \frac{1}{sL}\right)V_2 &= 0 \end{aligned}$$

However, this creates a problem for computerized solutions. Multiplication by  $s$  represents differentiation in the Laplace transform and  $1/s$  represents integration. As a result, the two equations are actually a set of two integro-differential equations, and we do not normally have methods to solve them directly in such a form. In all computerized methods, we use integration of systems of first-order differential equations and the preceding equations cannot be arranged into such a form. What we need is a method which will keep all frequency-dependent elements in the form  $sC$  or  $sL$ , with the variable  $s$  in the numerator. Such possibility exists if we take into account a new variable, the current through the inductor. Writing KCL for the two nodes of [Figure 6.8](#)

$$\begin{aligned} (G_1 + sC_1)V_1 + I_L &= J \\ (G_2 + sC_2)V_2 - I_L &= 0 \end{aligned}$$

We now have two equations but three variables. What we have not used yet is an expression which couples the voltages across the inductor with the current through it. The relationship is  $V_1 - V_2 = sLI_L$ , but because we do not know any of these three variables, we transfer everything to the left and write the last equation

$$V_1 - V_2 - sLI_L = 0 \tag{6.5}$$

All three can be put into a matrix form

$$\begin{bmatrix} G_1 + sC_1 & 0 & 1 \\ 0 & G_2 + sC_2 & -1 \\ 1 & -1 & -sL \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_L \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}$$

In this equation, the nodal portion is the  $2 \times 2$  matrix in the upper left corner and information about the inductor is collected in the right-most column and the lowest row. A larger network will have a larger nodal portion, but we still increase the matrix by one row and column for each inductor. This can be prepared as a general *stamp* as shown in [Figure 6.9](#). The previous matrix is the empty box, separated by the dashed lines, the voltages and the current above the stamp indicate the variables relevant to the inductor, while on the left the letters  $i$  and  $j$  give the rows (node numbers) where  $L$  is connected. Should we have a network with two inductors, we add one row and one column for each.

The next element we take is the voltage-controlled current source; it was already mentioned in [Section 6.2](#) as an element which can be taken into the nodal formulation. The VC is shown in [Figure 6.10](#).

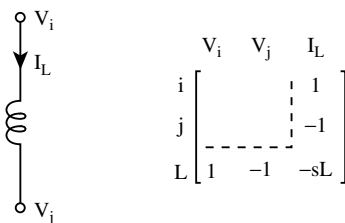


FIGURE 6.9 Stamp of an inductor.

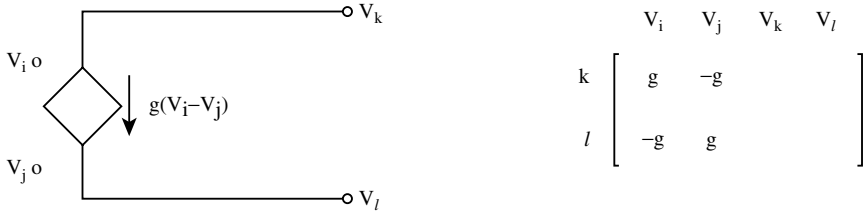


FIGURE 6.10 Stamp for a voltage-controlled current source.

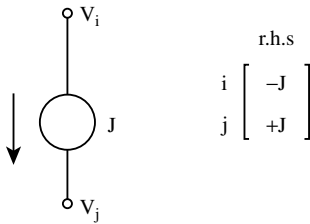


FIGURE 6.11 Stamp for an independent current source.

It adds the current  $g(V_i - V_j)$  to node  $k$  and subtracts the same current at node  $l$ . It permits us to write the stamp, Figure 6.10. The element influences the balance of currents at node  $i$  and  $j$ , and the variables which multiply its transconductance are  $V_i$  and  $V_j$ .

Network theory defines two independent sources: the current and the voltage source. The current source can be taken into consideration in the right-hand side (r.h.s.) of the nodal portion; its stamp is in Figure 6.11. The independent voltage source cannot be taken directly and we must add its current as a new variable. Consider the source with a resistor in series, shown in Figure 6.12. The voltage relationships are

$$V_i - V_j - rI_E = E \tag{6.6}$$

The current  $I_E$  adds to the balance of currents at node  $i$ , because it flows away from it. It is subtracted from the balance of currents at node  $j$ . The current is taken as a new variable and the equation is attached to previous equations. The stamp is as shown. It is, in fact, a combined stamp for several elements. If we set  $r = 0$ , we have an ideal voltage source. If we set both  $r$  and  $E$  equal to zero, we have a stamp for a short circuit. For better understanding, consider the example in Figure 6.13. The network has two ungrounded nodes for which we can write the nodal equations:

$$\begin{aligned} (G_1 + sC_1)V_1 + I_E &= 0 \\ (G_2 + sC_2)V_2 - I_E &= 0 \end{aligned}$$

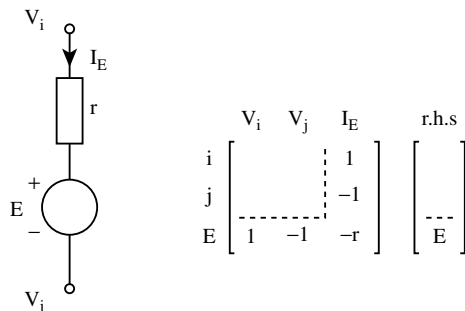


FIGURE 6.12 Stamp for an independent voltage source. The resistor value can be zero.

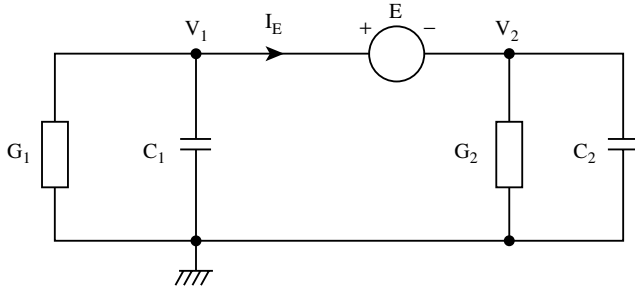


FIGURE 6.13 Example with a floating voltage source.

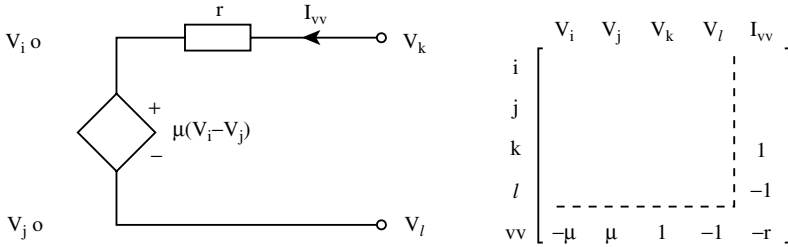


FIGURE 6.14 Stamp for a voltage-controlled voltage source. The resistor value can be zero.

To these equations, we must add the equation describing the properties of the ideal voltage source,

$$V_i - V_j = E \tag{6.7}$$

Collecting into matrix form

$$\begin{bmatrix} G_1 + sC_1 & 0 & 1 \\ 0 & G_2 + sC_2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix}$$

We still need stamps for the remaining three dependent sources. The voltage controlled voltage source,  $VV$ , is shown in Figure 6.14. For generality, we added the internal resistor. The output is described by the equation

$$\mu(V_i - V_j) + rI_{VV} = V_k - V_l \tag{6.8}$$

None of the voltages is known, so we transfer everything to the left side. Input terminals do not influence the balance of currents, but the output terminals do. At node  $k$  we must add  $I_{VV}$ , and subtract the same at node  $l$ . The stamp is in Figure 6.14.

The current controlled current source,  $CC$ , is shown in Figure 6.15. The input terminals are short circuited and

$$V_i - V_j = 0 \tag{6.9}$$

No information is available about the output voltages, but we know that the current is  $\alpha$  times the input current, and thus only one additional variable is needed. Balances of currents are influenced at all four nodes: positive  $I$  at node  $i$ , negative at node  $j$ , positive current  $\alpha I$  at node  $k$ , and negative  $\alpha I$  at node  $l$ .

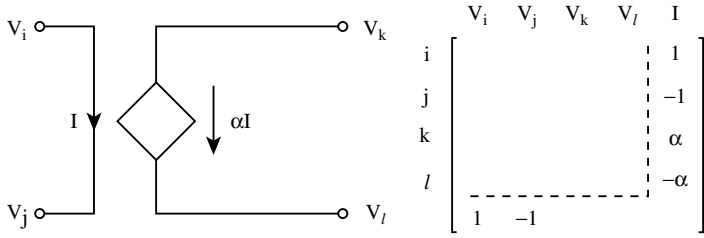


FIGURE 6.15 Stamp for a current-controlled current source.

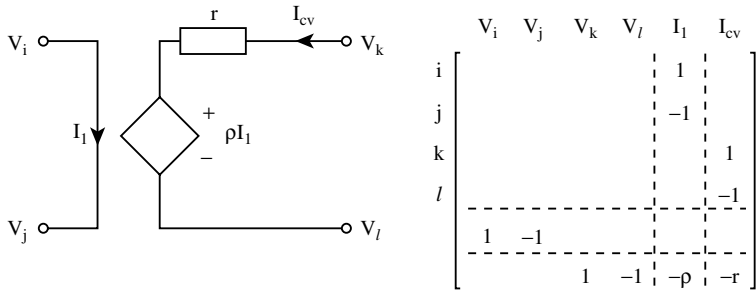


FIGURE 6.16 Stamp for a current-controlled voltage source. The resistor value can be zero.

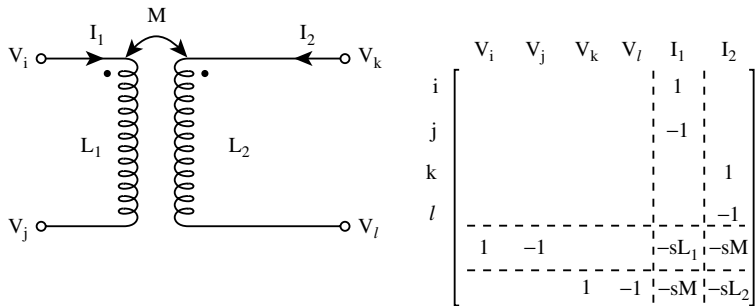


FIGURE 6.17 Stamp for a transformer.

The stamp is in Figure 6.15, where the added column takes care of the currents and the additional row describes the properties of the short circuit.

The most complicated dependent source is the current-controlled voltage source, *CV*, shown in Figure 6.16. We can consider it as a combination of a short circuit and a voltage source. The equation for the short circuit is the same as for the *CC*. The output is defined by the equation

$$V_k - V_l = \rho I_1 + r I_{CV} \tag{6.10}$$

and because none of the variables is known, we transfer everything to the left. This element adds two rows and two columns to the previously defined matrix. Its stamp is in Figure 6.16. As before, the internal resistor *r* can be set equal to zero.

Modified nodal formulation easily takes into account a transformer (see Figure 6.17). It is described by the equations

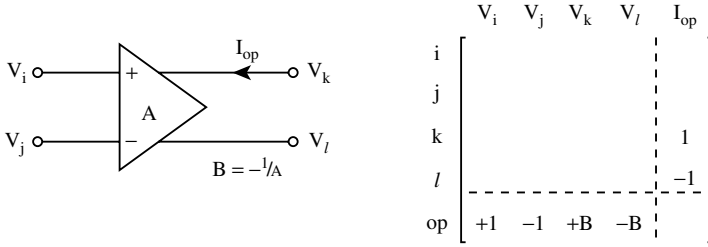


FIGURE 6.18 Stamp for an ideal operational amplifier.

$$\begin{aligned}
 V_i - V_j &= sL_1 I_1 + sM I_2 \\
 V_k - V_l &= sL_1 I_1 + sL_2 I_2
 \end{aligned}
 \tag{6.11}$$

None of the variables is known, and we transfer everything to the left. The currents influence the balance of currents at all nodes. The stamp of the transformer is in Figure 6.17.

The last element we consider is an ideal operational amplifier. It is sometimes taken as a voltage-controlled voltage source with very high gain, but it is preferable to have a stamp that can take into account ideal properties as well. The element is shown in Figure 6.18. The terminal  $l$  is usually grounded, but we will keep it floating to make the stamp more general. No current flows into the device at the input terminals. The output equation is

$$V_k - V_l = A(V_i - V_j)
 \tag{6.12}$$

Because a computer cannot handle infinity, it is advantageous to introduce the inverted gain

$$B = -1/A
 \tag{6.13}$$

and modify the previous equation to

$$V_i - V_j + B V_k - B V_l = 0
 \tag{6.14}$$

This equation is attached to the set of equations and the balance of currents is influenced at nodes  $k$  and  $l$ . This leads to the stamp in Figure 6.18. If we set  $B = 0$ , the operational amplifier becomes ideal, with no approximation.

An example will show how the stamps are used. Consider the network in Figure 6.19. It has no practical application, but serves well for the demonstration of how to set up the modified nodal matrix. A short circuit, indicating the controlling current of the current-controlled voltage source is taken into account

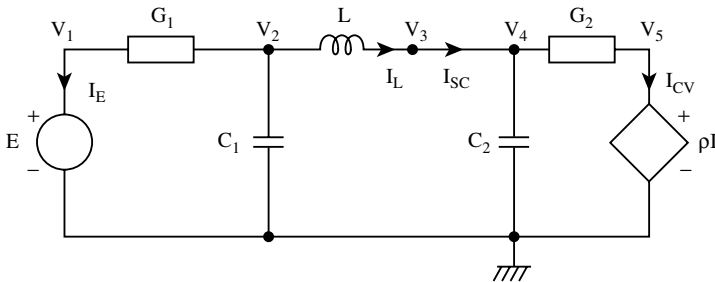


FIGURE 6.19 Example showing the use of modified nodal formulation.

by increasing the number of nodes. The network has five nongrounded nodes, and thus the dimension of its nodal portion will be 5. The voltage source will increase the system matrix by one row and one column, the inductor also, and the CV will need two more rows and columns; altogether the matrix will be  $9 \times 9$ . Write first the nodal portion by disregarding entirely the other elements. This creates the upper left partition. Using the stamps we add first the voltage source, then the inductor, next the short circuit, and finally the current-controlled voltage source. The system matrix is

$$\begin{bmatrix}
 G_1 & -G_1 & 0 & 0 & 0 & | & 1 & | & 0 & | & 0 & | & 0 \\
 -G_1 & G_1 + sC_1 & 0 & 0 & 0 & | & 0 & | & 1 & | & 0 & | & 0 \\
 0 & 0 & 0 & 0 & 0 & | & 0 & | & -1 & | & 1 & | & 0 \\
 0 & 0 & 0 & sC_2 + G_2 & -G_2 & | & 0 & | & 0 & | & -1 & | & 0 \\
 0 & 0 & 0 & -G_2 & G_2 & | & 0 & | & 0 & | & 0 & | & 1 \\
 \hline
 1 & 0 & 0 & 0 & 0 & | & 0 & | & 0 & | & 0 & | & 0 \\
 \hline
 0 & 1 & -1 & 0 & 0 & | & 0 & | & -sL & | & 0 & | & 0 \\
 \hline
 0 & 0 & 1 & -1 & 0 & | & 0 & | & 0 & | & 0 & | & 0 \\
 \hline
 0 & 0 & 0 & 0 & 1 & | & 0 & | & 0 & | & -\rho & | & 0
 \end{bmatrix}
 \begin{bmatrix}
 V_1 \\
 V_2 \\
 V_3 \\
 V_4 \\
 V_5 \\
 I_E \\
 I_L \\
 I_{SC} \\
 I_{CV}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 E \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Modified nodal formulation is the most important method for computer applications. The reader can find additional information in the books [1, 2].

### 6.5 Nonlinear Elements

In previous sections, we used the Laplace transform to explain the various methods of formulation. Because we dealt with linear elements, the systems of equations were linear and it was possible to cast them into matrix forms.

If we must consider nonlinear elements, we face many restrictions. The Laplace transform cannot be used. Various concepts based on it, like the network functions, the poles and the zeros, cannot be applied. Only two types of analysis are available:

- The dc solution (operating point)
- Time-domain solution for a given input signal

Once we have nonlinear elements, we cannot write the equations in matrix form; all we can do is write KCL equations. We must also find another method for the solution of such nonlinear equations. Consider two differentiable equations in two unknowns

$$\begin{aligned}
 f_1(v_1, v_2) &= 0 \\
 f_2(v_1, v_2) &= 0
 \end{aligned}
 \tag{6.15}$$

The functions can be expanded into Taylor series and the series truncated after the linear terms:

$$\begin{aligned}
 f_1(v_1 + \Delta v_1, v_2 + \Delta v_2) &= f_1(v_1, v_2) + \frac{\partial f_1}{\partial v_1} \Delta v_1 + \frac{\partial f_1}{\partial v_2} \Delta v_2 + \dots = 0 \\
 f_2(v_1 + \Delta v_1, v_2 + \Delta v_2) &= f_2(v_1, v_2) + \frac{\partial f_2}{\partial v_1} \Delta v_1 + \frac{\partial f_2}{\partial v_2} \Delta v_2 + \dots = 0
 \end{aligned}$$

We are not considering higher-order terms, so the equation will not be exactly zero, but we can still try to find  $\Delta v_1$  and  $\Delta v_2$ . Transferring the known function values to the right

$$\begin{aligned}\frac{\partial f_1}{\partial v_1} \Delta v_1 + \frac{\partial f_1}{\partial v_2} \Delta v_2 &= -f_1(v_1, v_2) \\ \frac{\partial f_2}{\partial v_1} \Delta v_1 + \frac{\partial f_2}{\partial v_2} \Delta v_2 &= -f_2(v_1, v_2)\end{aligned}$$

This is a system of *linear* equations, and we can rewrite it in matrix form

$$\begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \end{bmatrix} \begin{bmatrix} \Delta v_1 \\ \Delta v_2 \end{bmatrix} = \begin{bmatrix} -f_1(v_1, v_2) \\ -f_2(v_1, v_2) \end{bmatrix} \quad (6.16)$$

The matrix on the left is called the *Jacobian*; on the right is the negative of the functions. Once this *linear* system is solved, we can get new values of the variables by writing

$$\begin{aligned}v_1^{(i+1)} &= v_1^{(i)} + \Delta v_1^{(i)} \\ v_2^{(i+1)} &= v_2^{(i)} + \Delta v_2^{(i)}\end{aligned} \quad (6.17)$$

In this equation, we added the superscript to indicate iteration. The process is repeated until all  $\Delta v_i$  become sufficiently small. This iterative method is usually referred to as the Newton–Raphson iteration and is written in the form

$$\begin{aligned}\mathbf{J}(\mathbf{v}^{(i)}) \Delta \mathbf{v}^{(i)} &= -\mathbf{f}(\mathbf{v}^{(i)}) \\ \mathbf{v}^{(i+1)} &= \mathbf{v}^{(i)} + \Delta \mathbf{v}^{(i)}\end{aligned} \quad (6.18)$$

Suppose that we now take the network in [Figure 6.3](#), consider the conductances  $G_1$  and  $G_3$  as linear and replace  $G_2$  by a nonlinear function

$$i = g(v_{el})$$

where  $V_{el}$  is the voltage across this element,

$$v_{el} = v_1 - v_2$$

and  $g$  represents a nonlinear function. The two KCL equations are

$$\begin{aligned}G_1 v_1 + g(v_{el}) - J &= 0 \\ -g(v_{el}) + G_3 v_2 &= 0\end{aligned}$$

For the Newton–Raphson equation, we need the derivatives with respect to  $v_1$  and  $v_2$ . Consider now only the nonlinear element. Using the chain rule of differentiation we can write

$$\frac{\partial g(v_{el})}{\partial v_1} = \frac{\partial g(v_{el})}{\partial v_{el}} \frac{\partial v_{el}}{\partial v_1} = + \frac{\partial g(v_{el})}{\partial v_{el}}$$

$$\frac{\partial g(v_{el})}{\partial v_2} = \frac{\partial g(v_{el})}{\partial v_{el}} \frac{\partial v_{el}}{\partial v_2} = - \frac{\partial g(v_{el})}{\partial v_{el}}$$

With these preliminary steps, we can now write the Newton–Raphson equation:

$$\begin{bmatrix} G_1 + \frac{\partial g(v_{el})}{\partial v_{el}} & -\frac{\partial g(v_{el})}{\partial v_{el}} \\ -\frac{\partial g(v_{el})}{\partial v_{el}} & G_3 + \frac{\partial g(v_{el})}{\partial v_{el}} \end{bmatrix} \begin{bmatrix} \Delta v_1 \\ \Delta v_2 \end{bmatrix} = - \begin{bmatrix} G_1 v_1 + g(v_{el}) - J \\ -g(v_{el}) + G_3 v_3 \end{bmatrix}$$

Comparing the Jacobian of the Newton–Raphson equation with the nodal formulation, we reach the important conclusion, valid for networks of any size:

1. Linear elements will be in the Jacobian in the same position as they were in the linear system matrix.
2. Nonlinear elements will have entries in the same positions as if they were linear; only their numerical values will be equal to the derivative  $\partial g/\partial v_{el}$ , evaluated with already available variables.

This conclusion will also be true for the other formulations, similar to the tableau or the modified nodal. So far, we considered only nonlinear resistive elements and the operating point.

Nonlinear storage elements (capacitors and inductors) contribute to the equations with their fluxes and charges. The current through the nonlinear capacitor is defined by

$$i_c = \frac{dq(v_c)}{dt} \quad (6.19)$$

where  $q(v_c)$  is the charge and  $v_c$  is the voltage across the capacitor. The voltage across the nonlinear inductor is given by

$$v_L = \frac{d\phi(i_L)}{dt} \quad (6.20)$$

with  $\phi$  denoting the flux.

Integration of systems with storage elements is always done by first replacing the derivative by a suitable algebraic expression and then solving the resulting nonlinear algebraic system by the Newton–Raphson method derived previously.

Many methods are available to replace the time domain derivatives by algebraic expressions. Books on numerical analysis usually describe the Runge–Kutta method. We mention it here because it is *not* suitable for solution of networks. There are several reasons for this, the main one being that the preferred modified nodal formulation does not lead to systems of differential, but rather to systems of algebraic-differential equations. Only two methods are widely used, the trapezoidal formula and a family of backward differentiation formulas (BDF). Among the BDFs, the simplest is the backward Euler, and we will base our explanations on this formula. It replaces the derivative by the difference of the previous and new value, divided by the step size,  $h$ ,

$$\frac{dq(v_c)}{dt} \sim \frac{q_{new}(v_c) - q_{old}}{h}$$

$$\frac{d\phi(i_L)}{dt} \sim \frac{\phi_{new}(i_L) - \phi_{old}}{h}$$



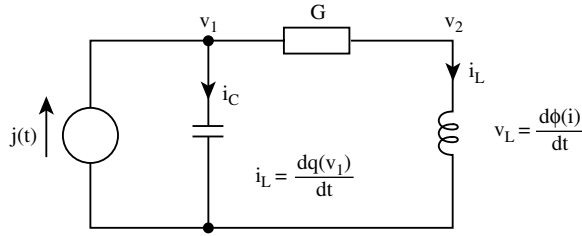


FIGURE 6.20 Network with a nonlinear capacitor and inductor.

Consider the network in Figure 6.20 with nonlinear storage elements and with a linear conductance,  $G$ . It can be described by three equations:

$$\begin{aligned}
 f_1 &= \frac{dq(v_c)}{dt} + G(v_1 - v_2) - j(t) = 0 \\
 f_2 &= -G(v_1 - v_2) + i_L = 0 \\
 f_3 &= v_2 - \frac{d\phi(i_L)}{dt} = 0
 \end{aligned}$$

The time derivatives are replaced by the backward Euler formula

$$\begin{aligned}
 \frac{q_{new}(v_1) - q_{old}}{h} + G(v_1 - v_2) - j(t) &= 0 \\
 -G(v_1 - v_2) + i_L &= 0 \\
 v_2 - \frac{\phi_{new}(i_L) - \phi_{old}}{h} &= 0
 \end{aligned}$$

thus changing the system into an algebraic one. If we now differentiate with respect to the variables  $v_1$ ,  $v_2$ , and  $i_L$ , we obtain the Jacobian

$$\begin{bmatrix}
 \frac{1}{h} \frac{\partial q_{new}(v_1)}{\partial v_1} + G & -G & 0 \\
 -G & +G & 1 \\
 0 & 1 & -\frac{1}{h} \frac{\partial \phi_{new}(i_L)}{\partial i}
 \end{bmatrix}$$

It can be observed that values of the derivatives are in the same places as would be the values of  $C(L)$  of linear capacitors (inductors). In addition, the variable  $s$  from the Laplace domain is replaced by  $1/h$ .

The example used a grounded capacitor and a grounded inductor. Figure 6.21 gives the stamps for floating nonlinear elements and for the Newton–Raphson iteration, based on the backward Euler formula.

## 6.6 Nodal Analysis of Active Networks

Low-frequency analog filters are often built with active RC networks and the active elements are almost always operational amplifiers. We have seen in Section 6.4 that each such element adds one row and one

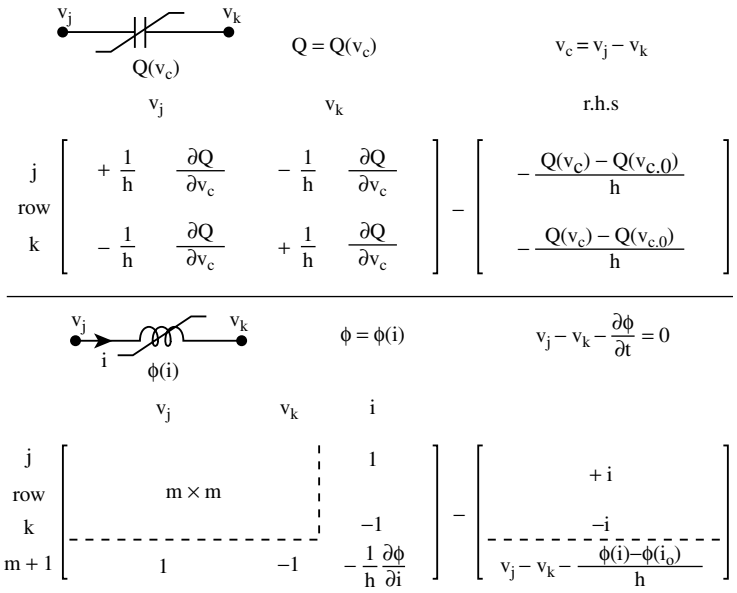


FIGURE 6.21 Stamp for a nonlinear capacitor and inductor.

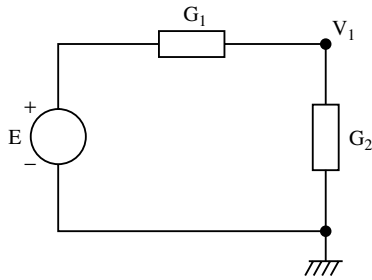


FIGURE 6.22 Example showing how to reduce the number of nodal equations.

column to the modified nodal system matrix, thus making the system too large for hand solutions. We need a method that can reduce the size of the matrix to the minimum. Such reduction is possible [1, 2], and becomes extremely simple if the voltage sources (dependent or independent) have one of their terminals grounded. Almost all practical networks meet this condition.

To introduce the method, consider the network in Figure 6.22. If we are not interested in the current through the voltage source, we can write only one nodal equation for the node on the right:

$$(G_1 + G_2) V_1 - EG_1 = 0$$

The source is known, the term in which it appears is transferred to the right side and, instead of three equations of the modified nodal formulation, we must solve only one. Consider next the network in Figure 6.23 with a voltage source and an ideal operational amplifier. One of the output terminals of the operational amplifier is grounded. Such amplifier is described by the equation

$$(V_+ - V_-)A = V_{out} \tag{6.21}$$

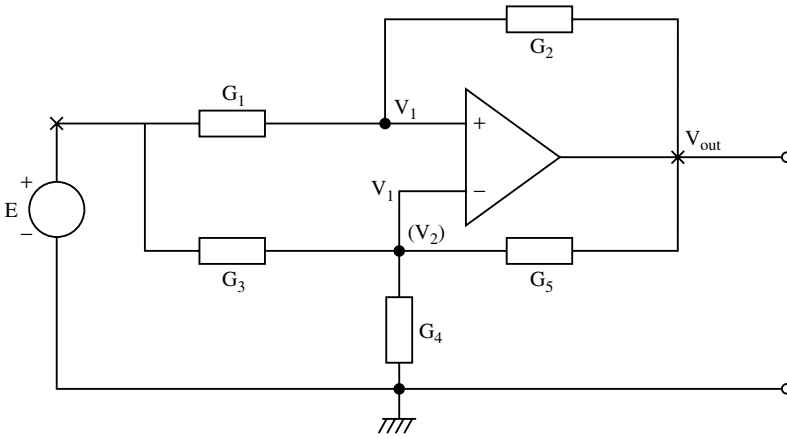


FIGURE 6.23 Nodal analysis of a network with one ideal operational amplifier.

where  $A \rightarrow \infty$ . Divide first by  $A$  and then substitute

$$B = -1/A \tag{6.22}$$

This changes the equation into

$$V_+ - V_- + BV_{out} = 0 \tag{6.23}$$

If the operational amplifier is ideal, set  $B = 0$  and in such case  $V_+ = V_-$ ; the operational amplifier will have the same voltages at its input terminals. We can take this into consideration, by simply writing the voltage with the same subscript to both input terminals of the operational amplifier, as was done in Figure 6.23. We are not interested in the current of the voltage source, nor in the current flowing into the operational amplifier. We mark our lack of interest by crossing out the nodes that have grounded voltage source; this was also done in Figure 6.23. Each node is given a voltage, but we write the nodal equations only at nodes that were not crossed out. For our example:

$$\begin{aligned} (G_1 + G_2)V_1 - G_2V_{out} - EG_1 &= 0 \\ (G_3 + G_4 + G_5)V_1 - G_5V_{out} - EG_3 &= 0 \end{aligned}$$

Terms with the known source voltage are transferred to the right and we have the system

$$\begin{bmatrix} G_1 + G_2 & -G \\ G_3 + G_4 + G_5 & -G_5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_{out} \end{bmatrix} = \begin{bmatrix} G_1E \\ G_3E \end{bmatrix}$$

A modified modal formulation would have required six equations.

This method can be used for any network if one node of each voltage source, dependent or independent, is grounded. All we have to do is assign every node a voltage, cross out nodes with voltage sources, and write nodal equations for the rest. It is advantageous to use conductances for resistors, because this way we avoid the fractions.

The method remains valid if the operational amplifier is not ideal and has the inverted gain  $B$ . The only difference is that for a nonideal amplifier we cannot make any assumptions on the voltages at its input terminals and the subscripts of such voltages must be different. This second case is also illustrated in Figure 6.23 by the voltage  $V_2$  (in brackets) at the lower node. We still write nodal equations for the

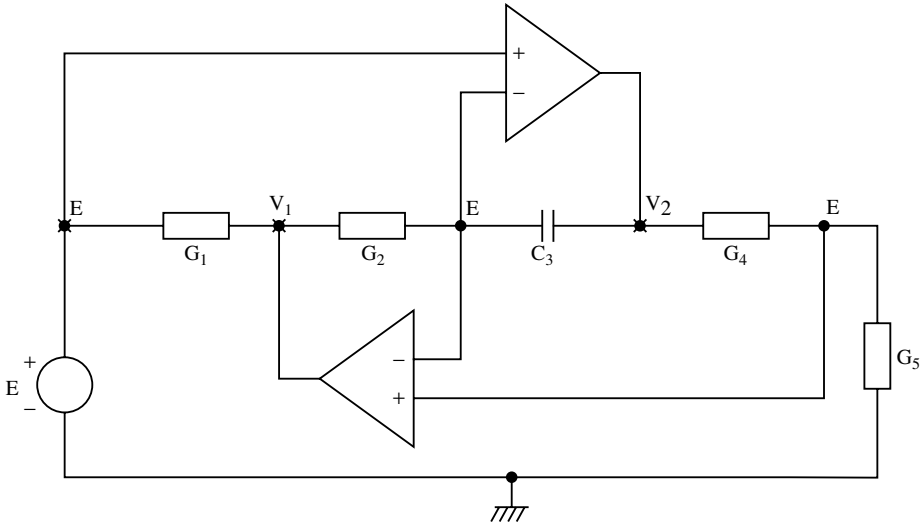


FIGURE 6.24 Nodal analysis of a network with two ideal operational amplifiers.

two input terminals, but to complete the system we must also attach the equation of the operational amplifier. The result is

$$\begin{aligned}
 V_1(G_1 + G_2) - G_2 V_{out} &= G_1 E \\
 V_1(G_3 + G_4 + G_5)V_2 - G_5 V_{out} &= G_3 E \\
 V_1 - V_2 + B V_{out} &= 0
 \end{aligned}$$

and in matrix from

$$\begin{bmatrix} G_1 + G_2 & 0 & -G_2 \\ 0 & G_3 + G_4 + G_5 & -G_5 \\ 1 & -1 & B \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_{out} \end{bmatrix} = \begin{bmatrix} G_1 E \\ G_3 E \\ 0 \end{bmatrix}$$

where  $B$  can be set zero for an ideal operational amplifier.

We will give one example of a practical network, Figure 6.24. The operational amplifiers are ideal and thus the input voltage,  $E$ , appears at three terminals of the network. The other terminal voltages are marked by  $V_1$  and  $V_2$ . The terminals with voltage sources are marked by crosses and only nodes 3 and 5, counting from left, remain for writing the KCL. They are

$$\begin{aligned}
 (G_2 + sC_3)E - G_2 V_1 - sC_3 V_2 &= 0 \\
 (G_4 + G_5)E - G_4 V_2 &= 0
 \end{aligned}$$

Transferring terms containing the independent voltage source,  $E$ , to the other side of the equation, we arrive at the system

$$\begin{bmatrix} G_2 & sC_3 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (G_2 + sC_3)E \\ (G_4 + G_5)E \end{bmatrix}$$

## **Conclusion**

It was demonstrated that hand calculations should use nodal or mesh formulations. Computer applications should be based on modified nodal formulation. For active networks, it is advantageous to use the method of Section 6.6.

## **References**

- [1] J. Vlach, *Basic Network Theory with Computer Applications*, New York: Van Nostrand Reinhold, 1992.
- [2] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*, 2nd ed., New York: Van Nostrand Reinhold, 1994.