7

Frequency Domain Methods

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7.1 Network Functions

Network functions are employed to characterize linear, time-invariant networks in the zero state for a single excitation. Network functions contain information concerning a network's stability and natural modes. They allow a designer to focus on obtaining a desired output signal for a given input signal.

In this section, it is shown that the concept of network functions is obtained as an extension of the (transformed) element defining equations for resistors, capacitors, and inductors. The relationships of network functions to transformed loop and node equations are also described. As a result of these relationships, a list of properties of network functions can be generated, which is useful in the analysis of linear networks. Much is known about a network function for a given network even before an analysis is performed and the function itself is obtained.

Ohm's law, $v_R(t) = Ri_R(\mathrm{t})$ where R is in ohms, describes the relationship between the voltage across the resistor and the current through the resistor. These variables and their reference polarity and direction are depicted in Figure 7.1. If elements of the equation for Ohm's law are transformed, we obtain $V(s) = RI(s)$ because R is a constant. Thus, we obtain an Ohm's law-like expression in the frequency domain.

However, the capacitor's voltage and current are related by the integral

$$\begin{aligned} v_c(t) &= \frac{1}{C}\int_{-\infty}^{t} i_c(t)dt = \frac{1}{C}\int_{-\infty}^{0} i_C(t)dt + \frac{1}{C}\int_{0}^{t} i_C(t)dt \\ &= V_0 + \frac{1}{C}\int_{0}^{t} i_c(t)dt \end{aligned} \tag{7.1}$$

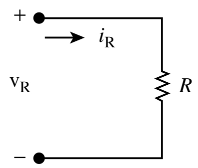


FIGURE 7.1 Reference polarity and direction for Ohm's law.

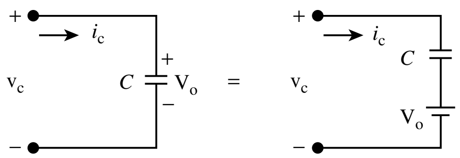


FIGURE 7.2 Capacitor representation showing reference polarities for voltages and reference direction for current.

where the voltage reference polarity and the current reference direction are illustrated in Figure 7.2 and the capacitance C is given in farads (F). Unlike the resistor, the relation between the capacitor voltage and the capacitor current is not a simple Ohm's law-like expression in the time domain. In addition, the voltage across the capacitor at any time t, is dependent on the entire history of the current through the capacitor.

The integral expression for the voltage across the capacitor can be split into two terms where the first term is the initial voltage across the capacitor, $V_0 = v_c(0)$. If the elements of the equation are transformed, we obtain

$$V_c(s) = \frac{V_0}{s} + \frac{1}{C}\frac{I_c(s)}{s} \tag{7.2}$$

Equation (7.2) shows that if $V_0 = 0$, then the expression for the transform of the capacitor voltage becomes more nearly Ohm's law-like in its form. Furthermore, if we associate the s that arises because of the integral of the current with the capacitor C, and define the *impedance* of the capacitor as $Z(s) = V_c(s)/I_c(s) = 1/(sC)$, then the equation becomes Ohm's law-like in the frequency domain.

A similar process can be applied to the inductor. The current through the inductor is expressed as

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt = I_0 + \frac{1}{L}\int_{0}^{t} v_L(t)dt \tag{7.3}$$

where L is expressed in henries (H) and $I_0 = i_L(0)$ is the initial current through the inductor. Figure 7.3 depicts the reference polarity and direction for the inductor voltage and current. If the expression for the current through the inductor is transformed, the result is:

$$I_L(s) = \frac{I_o}{s} + \frac{1}{L}\frac{V_L(s)}{s} \tag{7.4}$$

Again, as with the capacitor, if $I_0 = 0$ and if the s that is included because of the integral of $v_L(t)$ is considered as associated with L, then the expression for the transform of the current through the inductor has an Ohm's law-like form if we define the *impedance* of the inductor as $Z(s) = V_L(s)/I_L(s) = sL$.

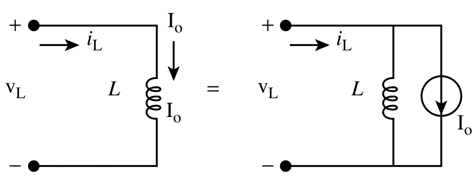


FIGURE 7.3 Inductor representation showing reference directions for currents and reference polarity for voltage.

The impedance concept is an important one in network analysis. It allows us to combine dissimilar elements in the frequency domain — something we cannot do in the time domain. In fact, impedance is a frequency domain concept. It is the ratio of the *transform* of the voltage across the port of the network to the *transform* of the current through the port with all independent sources within the network properly removed and with all initial voltages across capacitors and initial currents through inductors set to zero. Thus, when we indicate that independent sources are to be removed, we mean that initial conditions are to be set to zero as well.

The concept of impedance can be extended to linear, lumped, finite, time-invariant, one-port networks in general. We denote these networks as LLFT networks. These networks are linear. That is, they are composed of elements including resistors, capacitors, inductors, transformers, and dependent sources with parameters that are not functions of the voltage across the element or the current through the element. Thus, the differential equations describing these networks are linear.

These networks are lumped and not distributed. That is, LLFT networks do not contain transmission lines as network elements, and the differential equations describing these networks are ordinary and not partial differential equations.

LLFT networks are finite, meaning that they do not contain infinite networks and require only a finite number of network elements in their representation. Infinite networks are sometimes useful in modeling such things as ground connections in the surface of the earth, but we exclude the discussion of them here.

LLFT networks are time-invariant or constant instead of time-varying. Thus, the ordinary, linear differential equations describing LLFT networks have constant coefficients.

The steps for finding the impedance of an LLFT one-port network are:

1. Properly remove all independent sources in the network. By "properly" removing independent sources, we mean that voltage sources are replaced by short circuits and current sources are replaced by open circuits. Dependent sources are not removed.
2. Excite the network with a voltage source or a current source at the port, and find an equation or equations to solve for the other port variable.
3. Form $Z(s) = V(s)/I(s)$.

Simple networks do not need to be excited in order to determine their impedance, but in the general case an excitation is required. The next example illustrates these concepts.

Example 1. Find the impedances of the one-port networks in Figure 7.4.

Solution. The network in Figure 7.4(a) is composed of three elements connected in series. No independent sources are present, and there is zero initial voltage across the capacitor and zero initial current through the inductor. The impedance is determined as:

$$Z(s) = R + sL + \frac{1}{sC} = L\left(\frac{s^2 + s\frac{R}{L} + \frac{1}{LC}}{s}\right)$$

The network in Figure 7.4(b) includes a dependent source that depends on the voltage across R_1. The impedance of this network is not obvious, and so we should excite the port. Also, the capacitor has an

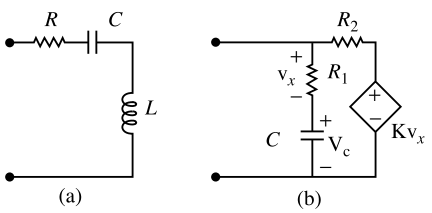


FIGURE 7.4 (a) A simple network. (b) A network containing a dependent source.

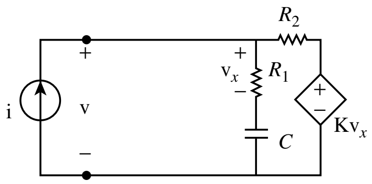


FIGURE 7.5 The network in Figure 7.4(b) prepared for analysis.

initial voltage V_c across it. This voltage is set to zero to find the impedance. Figure 7.5 is the network in Figure 7.4(b) prepared for finding the impedance. Using the impedance concept and two loop equations or two node equations, we obtain:

$$Z(s) = \frac{V(s)}{I(s)} = \frac{R_1 R_2\left(s + \frac{1}{CR_1}\right)}{\left[R_1(1-K)+R_2\right]\left(s + \frac{1}{C\left[R_1(1-K)+R_2\right]}\right)}$$

□

The expressions for impedance found in the previous example are rational functions of s, and the coefficients of s are functions of the elements of the network including the coefficient K of the dependent source in the network in Figure 7.4(b). We will demonstrate that these observations are general for LLFT networks; but first we will extend the impedance concept in another direction.

We have defined the impedance of a one-port LLFT network. We can also define another network function — the admittance $Y(s)$. The admittance of a one-port LLFT network is the quotient of the *transform* of the current through the port to the *transform* of the voltage across the port with all independent sources within the network properly removed. One-port networks have only two linear network functions, impedance and admittance. Furthermore, $Z(s) = 1/Y(s)$ because both network functions concern the same port of the network, and the impedance or admittance relating the response to the excitation is the same whether a current excitation causes a voltage response or a voltage excitation causes a current response. An additional implication of this observation is that either network function can be determined with either type of excitation, voltage source or current source, applied to the network.

Figure 7.6 depicts a two-port network with the reference polarities and reference directions indicated for the port variables. Port one of the two-port network is formed from the two terminals labeled 1 and 1′. The two terminals labeled 2 and 2′ are associated to form port two. A two-port network has 12 network functions associated with it instead of only two, and so we will employ the following notation for these functions:

$$N_{RE}(s) = R(s)/E(s)$$

where $N_{RE}(s)$ is a network function, the subscript "R" is the port at which the response variable exists, the subscript "E" is the port at which the excitation is applied, $R(s)$ is the transform of the response variable, and $E(s)$ is the transform of the excitation that may be a current source or a voltage source depending on the particular network function. For example, for the two-port networks shown in Figure 7.7

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} \text{ and } G_{12}(s) = \frac{V_1(s)}{V_2(s)} \tag{7.5}$$

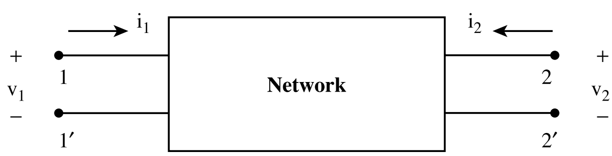


FIGURE 7.6 Reference polarities and reference directions for port variables of a two-port network.

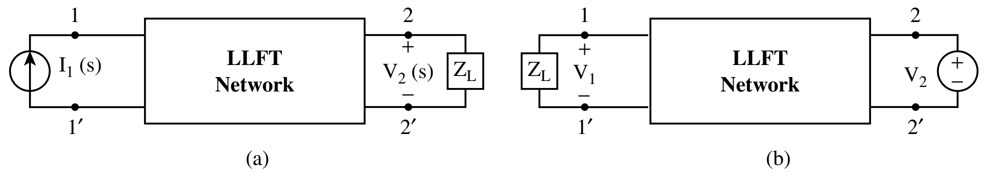


FIGURE 7.7 (a) Network configured for finding $Z_{21}(s)$. (b) Network for determining $G_{12}(s)$.

TABLE 7.1 Network Functions of Two-Port Networks

	Excitation Port	
Response Port	1	2
1	Z_{11}, Y_{11}	Z_{12}, Y_{12}, G_{12}, α_{12}
2	$Z_{21,}$ Y_{21}, $G_{21,}$ α_{21}	Z_{22}, Y_{22}

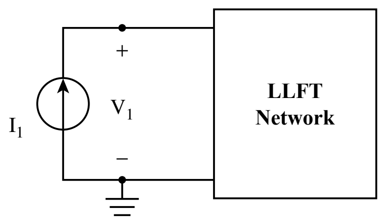


FIGURE 7.8 An LLFT network with n independent nodes plus the ground node.

Note that a load impedance has been placed across port two, the response port for $Z_{21}(s)$, in Figure 7.7(a). Also, a load has been connected across port one, the response port for $G_{12}(s) = V_1(s)/V_2(s)$ in Figure 7.7(b). It is assumed that all independent sources have been properly removed in both networks in Figure 7.7, and this assumption also applies to the loads. Of course, if a load impedance is changed, usually the network function will change. Thus, the load, if any, must be specified.

Table 7.1 lists the network functions of a two-port network. "G" denotes a voltage ratio, and "α" denotes a current ratio. The functions can also be grouped into driving-point and transfer functions. Driving-point functions are ones in which the excitation and response occur at the same port, and transfer network functions are ones in which the excitation and response occur at different ports. For example, $Z_{11}(s) = V_1(s)/I_1(s)$ is a driving-point network function, and $G_{21}(s) = V_2(s)/V_1(s)$ is a transfer network function. Of the twelve network functions for a two-port network, four are driving-point functions and eight are transfer functions. The two network functions for a one-port network are, of necessity, driving-point network functions.

Network functions are related to loop and node equations. Consider the LLFT network in Figure 7.8. Independent sources within the network have been properly removed. Assume the network has n independent nodes plus the ground node, and assume for simplicity that the network has no mutual inductance or dependent sources. Let us determine $Z_{11} = V_1/I_1$ in terms of a quotient of determinants of the nodal admittance matrix. The node equations, all written with currents leaving a node as positive currents, are

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.6}$$

where V_i, $i = 1,2,\ldots, n$, are the unknown node voltages. The elements y_{ij}, $i, j = 1,2,\ldots, n$, of the nodal admittance matrix above have the form

$$y_{ij} = \pm\left\{g_{ij} + \frac{\Gamma_{ij}}{s} + sC_{ij}\right\} \tag{7.7}$$

where the plus sign is taken if $i = j$ and the minus sign is taken if i and j are unequal. The quantity g_{ij} is the sum of the conductances connected to node i if $i = j$, and if i does not equal j, it is the sum of the conductances connected between nodes i and j. A similar statement applies to C_{ij}. The quantity Γ_{ij} is the sum of the reciprocal inductances ($\Gamma = 1/L$) connected to node i if $i = j$, and it is the sum of the reciprocal inductances connected between nodes i and j if i does not equal j.

Solving for V_1 using Cramer's rule yields:

$$V_1 = \frac{\begin{vmatrix} I_1 & y_{12} & \cdots & y_{1n} \\ 0 & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_{n2} & \cdots & y_{nn} \end{vmatrix}}{\Delta'} \tag{7.8}$$

where Δ' is the determinant of the nodal admittance matrix. Thus,

$$V_1 = I_1 \frac{\Delta'_{11}}{\Delta'} \tag{7.9}$$

where Δ'_{11} is the cofactor of element y_{11} of the nodal admittance matrix. Thus, we can write

$$\frac{V_1}{I_1} = Z_{11} = \frac{\Delta'_{11}}{\Delta'} \tag{7.10}$$

If mutual inductance and dependent sources exist in the LLFT network, the nodal admittance matrix elements are modified. Furthermore, there may be more than one entry in the column matrix containing excitations. However, Z_{11} can still be expressed as a quotient of determinants of the nodal admittance matrix.

Next, consider the network in Figure 7.9, which is assumed to have n independent nodes plus a ground node. The response port exists between terminals j and k. In this network, we are making the pair of terminals j and k serve as the second port. Denote the **transimpedance** V_{jk}/I_1 as Z_{j1}. Let us express this transfer function Z_{j1} as a quotient of determinants. All independent sources within the network have been properly removed. Note that the node voltages are measured with respect to the ground terminal indicated, but the output voltage is the difference of the node voltages V_j and V_k. Thus, we have to solve

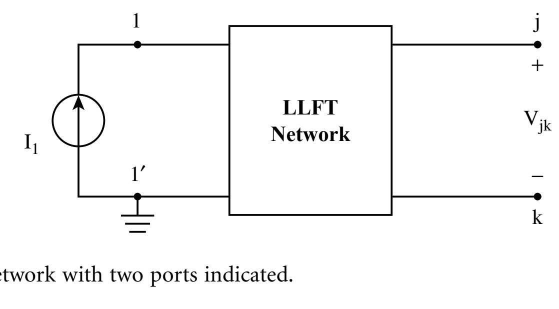


FIGURE 7.9 An LLFT network with two ports indicated.

for these two node voltages. Writing node equations, again taking currents leaving a node as positive currents, and solving for V_j using Cramer's rule we have

$$V_j = \frac{\begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1(j-1)} & I_1 & y_{1k} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2(j-1)} & 0 & y_{2k} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{n(j-1)} & 0 & y_{nk} & \cdots & y_{nn} \end{vmatrix}}{\Delta'} \tag{7.11}$$

which can be written as

$$V_j = I_1 \frac{\Delta'_{1j}}{\Delta'} \tag{7.12}$$

where Δ'_{1j} is the cofactor of element y_{1j} of the nodal admittance matrix. Similarly, we can solve for V_k and obtain

$$V_k = I_1 \frac{\Delta'_{1k}}{\Delta'} \tag{7.13}$$

Then, the transimpedance Z_{j1} can be expressed as

$$Z_{j1} = \frac{V_j - V_k}{I_1} = \frac{\Delta'_{1j} - \Delta'_{1k}}{\Delta'} \tag{7.14}$$

If terminal k in Figure 7.9 is common with the ground node so that the network is a grounded two-port network, then V_k is zero, and Z_{j1} can be expressed as Δ'_{1j}/Δ'. This result can be extended so that if the output voltage is taken between any node h and ground, then the transimpedance can be expressed as $Z_{h1} = \Delta'_{1h}/\Delta'$.

These results can be used to obtain an expression for G_{21} in terms of the determinants of the nodal admittance matrix. Figure 7.10 shows an LLFT network with a voltage excitation applied at port 1 and with port 2 open. The current $I_1(s)$ is given by V_1/Z_{11}. Then, V_2 is given by $V_2(s) = I_1 Z_{21}$. Thus,

$$G_{21} = \frac{V_2}{V_1} = \frac{I_1 Z_{21}}{I_1 Z_{11}} = \frac{\Delta'_{12}}{\Delta'_{11}} \tag{7.15}$$

Note that the determinants in the quotient are of equal order so that G_{21} is dimensionless.

Of course, network functions can also be expressed in terms of determinants of the loop impedance matrix. Consider the two-port network in Figure 7.11, which is excited with a voltage source applied to

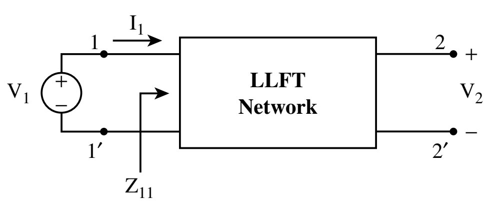


FIGURE 7.10 An LLFT network with port 2 open.

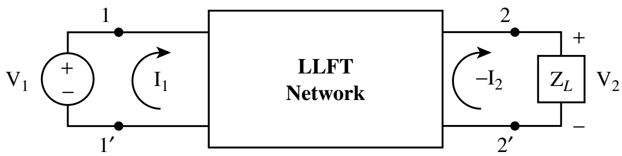


FIGURE 7.11 An LLFT network with load Z_L connected across port 2.

port 1 and has a load Z_L connected across port 2. Let us find the voltage transfer function G_{21} using loop equations. Assume that n independent loops exist, of which two are illustrated explicitly in Figure 7.11, and assume that no independent sources are present within the network. Also, assume for simplicity that the network contains no dependent sources or mutual inductance and that the loops are chosen so that V_1 is in only one loop. The loop equations are:

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ -I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.16}$$

where I_j, $j = 1,3,\ldots,n$, and $-I_2$ are the loop currents, and the elements z_{ij} of the loop impedance matrix are given by:

$$z_{ij} = \pm\left(R_{ij} + sL_{ij} + \frac{D_{ij}}{s}\right), \quad i, j = 1, 2, \ldots, n \tag{7.17}$$

where we have assumed that all loop currents are taken in the same direction such as clockwise. The plus sign applies if $i = j$, and the minus sign is used if $i \neq j$. R_{ij} is the sum of the resistances in loop i if $i = j$, and R_{ij} is the sum of the resistances common to loops i and j if $i \neq j$. L_{ij} is the sum of the inductances in loop i if $i = j$, and it is the sum of the inductances common to loops i and j if $i \neq j$. A similar statement applies to the reciprocal capacitances $D_{ij}(D = 1/C)$. However, the element z_{22} includes the extra term Z_L which could be a quotient of determinants itself. Solving for $-I_2$ using Cramer's rule, we have Δ, which is the determinant of the $n \times n$ loop impedance matrix, and Δ_{12} is the cofactor of element z_{12} of the loop impedance matrix.

$$-I_2 = \frac{\begin{vmatrix} z_{11} & V_1 & z_{13} & \cdots & z_{1n} \\ z_{21} & 0 & z_{23} & \cdots & z_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n1} & 0 & z_{n3} & \cdots & z_{nn} \end{vmatrix}}{\Delta} = V_1 \frac{\Delta_{12}}{\Delta} \tag{7.18}$$

The transform voltage V_2 is given by $-I_2(s)Z_L$, and the transfer function G_{21} can be expressed as

$$G_{21} = \frac{V_2}{V_1} = Z_L \frac{\Delta_{12}}{\Delta} \tag{7.19}$$

Thus, G_{21} can be represented as a quotient of determinants of the loop impedance matrix multiplied by Z_L.

In a similar manner, we can write

$$Y_{jk} = \frac{\Delta_{kj}}{\Delta}, \quad j,k = 1,2 \tag{7.20}$$

Then, we can use this result to write:

$$V_1 = I_1 \frac{1}{Y_{11}} \text{ and } I_2 = V_1 Y_{21} \tag{7.21}$$

Thus,

$$\frac{I_2}{I_1} = \alpha_{21} = \frac{\Delta_{12}}{\Delta_{11}} \tag{7.22}$$

Table 7.2 summarizes some of these results.

TABLE 7.2 Network Function in Terms of Quotients of Determinants

$Y_{jk} = \Delta_{kj}/\Delta$	$G_{jk} = \Delta'_{kj}/\Delta'_{kk}$
$Z_{jk} = \Delta'_{kj}/\Delta'$	$\alpha_{jk} = \Delta_{kj}/\Delta_{kk}$

Properties of LLFT Network Functions

We now list properties of network functions of LLFT networks. These properties are useful as a check of an analysis of such networks.

1. Network functions of LLFT networks are real, rational functions of *s*, and therefore have the form

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_{m-1} s + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n} \tag{7.23}$$

where the coefficients in both the numerator and denominator polynomials are real.

A network function of an LLFT network is a rational function because network functions can be expressed as quotients of determinants of nodal admittance matrices or of loop impedance matrices. The elements of these determinants are at most simple rational functions, and when the determinants are expanded and the fractions cleared, the result is always a rational function. The coefficients a_i, $i = 0,1,\ldots, m$, and b_j, $j = 0,1,\ldots, n$, are functions of the real elements of the network R, L, C, M, and coefficients of dependent sources. The constants R, L, C, and M are real. In most networks, the coefficients of dependent sources are real constants. Thus, the coefficients of LLFT network functions are real, and therefore the network function is a real function. It is possible for the "coefficients" of dependent sources in LLFT networks to themselves be real, rational functions of *s*. But when all the fractions are cleared, the result is a real, rational function.

2. An LLFT network function is completely defined by its self-poles, self-zeros, and the scale factor H.

If the numerator and denominator polynomials are factored and there are no common factors, we have

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0}{b_0} \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \tag{7.24}$$

where a_0/b_0 = H is the scale factor. The values of $s = z_1, z_2, \ldots, z_m$ are zeros of the polynomial $P(s)$ and self-zeros of the network function. Also, $s = p_1, p_2, \ldots, p_n$ are zeros of the polynomial $Q(s)$ and self-poles of the network function. In addition, $N(s)$ may have other poles or zeros at infinity. We call these poles and zeros **mutual poles** and **mutual** zeroes because they result from the difference in degrees of the numerator and denominator polynomials.

3. Counting both self and mutual poles and zeros, and counting a k^{th} order pole or zero k times, N(s) has the same number of poles as zeros, and this number is equal to the highest power of s in $N(s)$.

If $m = n$, then there are n self-zeros and n self-poles and no mutual poles or zeros. If $n > m$, then there are m self-zeros and n self-poles. There are also $n - m$ mutual zeros. Thus, there are n poles and $m + (n - m) = n$ zeros. A similar statement can be constructed for $n < m$.

4. Complex roots of $P(s)$ and $Q(s)$ occur in conjugate pairs.

This property follows from the fact that the coefficients of the numerator and denominator polynomials are real. Thus, complex factors of these polynomials have the form

$$(s+c+jd)(s+c-jd)=\left[(s+c)^2+d^2\right] \tag{7.25}$$

where c and d are real constants.

5. A driving point function of a network having no dependent sources can have neither poles nor zeros in the right-half s-plane (RHP), and poles and zeros on the imaginary axis must be simple. The same restrictions apply to the poles of transfer network functions of such networks but not to the zeros of transfer network functions.

Elsewhere in this handbook it is shown that the denominator polynomials of LLFT networks having no dependent sources cannot have RHP roots, and roots on the imaginary axis, if any, must be simple. However, the reciprocal of a driving-point network function is also a network function. For example, $1/Y_{22} = Z_{22}$. Thus, restrictions on the locations of poles of driving-point network functions also apply to zeros of driving-point network functions.

However, the reciprocal of a transfer network function is not a network function (see [5]). For example, $1/Y_{21} \neq Z_{21}$. Thus, restrictions on the poles of a transfer function do not apply to its zeros.

We can make a classification of the factors corresponding to the allowed types of poles as follows:

TABLE 7.3 A Classification of Factors of Network Functions of *LLFT* Networks Containing No Dependent Sources

Type	Factor(s)	Conditions
A	$(s + a)$	$a \geq 0$
B	$(s + b + jc)(s + b - jc)$	$b > 0, c > 0$
C	$(s + jd)(s - jd)$	$d > 0$

The Type A factor corresponds to a pole on the $-\sigma$ axis. If $a = 0$, then the factor corresponds to a pole on the imaginary axis, and so only one such factor is allowed. Type B factors correspond to poles in the left-half s-plane (LHP), and Type C factors correspond to poles on the imaginary axis.

6. The coefficients of the numerator and denominator polynomials of a driving-point network function of an LLFT network with no dependent sources are positive. The coefficients of the denominator polynomial of a transfer network function are all one sign. Without loss of generality, we take the sign to be positive. But some or all of the coefficients of the numerator polynomial of a transfer network function may be negative.

A polynomial made up of the factors listed in Table 7.3 would have the form:

$$Q(s)=(s+a_1)\cdots\left[(s+b_1)^2+c_1^2\right]\cdots(s^2+d_1^2)\cdots$$

Note that all the constants are positive in the expression for $Q(s)$, and so it is impossible for any of the coefficients of $Q(s)$ to be negative.

7. There are no missing powers of s in the numerator and denominator polynomials of a driving-point network function of an LLFT network with no dependent sources unless all even or all odd powers of s are missing or the constant term is missing. This statement holds for the denominator polynomials of transfer functions of such networks, but there may be missing powers of s in the numerator polynomials of transfer functions.

Property 7 is easily illustrated by combining types of factors from Table 7.3. Thus, a polynomial consisting only of type A factors contains all powers of s between the highest power and the constant term unless one of the "a" constants is zero. Then, the constant term is missing. Two a constants cannot be zero because then there would be two roots on the imaginary axis at the same location. The roots on the imaginary axis would not be simple.

A polynomial made up of only type B factors contains all powers of s, and a polynomial containing only type C factors contains only even powers of s. A polynomial constructed from type C factors except for one A type factor with $a = 0$ contains only odd powers of s. If a polynomial is constructed from type B and C factors, then it contains all power of s.

8. The orders of the numerator and denominator polynomials of a driving-point network function of an LLFT network, which contains no dependent sources can differ by no more than one.

The limiting behavior at high frequency must be that of an inductor, a resistor, or a capacitor. That is, if $N_{dp}(s)$ is a driving-point network function, then

$$\lim_{s\to\infty} N_{dp}(s) = \begin{cases} K_1 s \\ K_2 \\ K_3/s \end{cases}$$

where K_i, = 1, 2, 3, are real constants.

9. The terms of lowest order in the numerator and denominator polynomials of a driving-point network function of an LLFT network containing no dependent sources can differ in order by no more than one.

The limiting behavior at low frequency must be that of an inductor, a resistor, or a capacitor. That is,

$$\lim_{s\to 0} N_{dp}(s) = \begin{cases} K_4 s \\ K_5 \\ K_6/s \end{cases}$$

where the constants K_i, $i = 4, 5, 6$, are real constants.

10. The maximum order of the numerator polynomials of the dimensionless transfer functions G_{12}, G_{21}, α_{12}, and α_{21}, of an LLFT network containing no dependent sources is equal to the order of the denominator polynomials. The maximum order of the numerator polynomial of the transfer functions Y_{12}, Y_{21}, Z_{12}, and Z_{21} is equal to the order of the denominator polynomial plus 1. However, the minimum order of the numerator polynomial of any transfer function may be zero.

If dependent sources are included in an LLFT network, then it is *possible* for the network to have poles in the RHP or multiple poles at locations on the imaginary axis. However, an important application of stable networks containing dependent sources is to mimic (simulate) the behavior of LLFT networks that contain no dependent sources. For example, networks that contain resistors, capacitors, and dependent sources can mimic the behavior of networks containing only resistors, capacitors, and inductors. Thus, low-frequency filters can be constructed without the need for heavy, expensive inductors that would ordinarily be required in such applications.

7.2 Network Theorems

In this section, we provide techniques, strategies, equivalences, and theorems for simplifying the analysis of LLFT networks or for checking the results of an analysis. They can save much work in the analysis of some networks if one remembers to apply them. Thus, it is convenient to have them listed in one place. To begin, we list nine equivalences that are often called source transformations.

Source Transformations

Table 7.4 is a collection of memory aids for the nine source transformations. Source transformations are simple ways the elements and sources externally connected to a network N can be combined or eliminated without changing the voltages and currents within network N thereby simplifying the problem of finding a voltage or current within N.

Source transformation one in Table 7.4 shows the equivalence between two voltage sources connected in series and a single voltage source having a value that is the sum of the voltages of the two sources. A double-headed arrow is shown between the two network representations because it is sometimes advantageous to use this source transformation in reverse. For example, if a voltage source that has both DC and AC components is applied to a linear network N, it may be useful to represent that voltage source as two voltage sources in series — one a DC source and the other an AC source.

Source transformation two shows two voltage sources connected in parallel. Unless V_1 and V_2 are equal, the network would not obey Kirchhoff's law as evidenced by a loop equation written in the loop formed by the two voltage sources. A network that does not obey Kirchhoff's laws is termed a contradiction. Thus, a single-headed arrow is shown between the two network representations.

Source transformations three and four are duals, respectively, of source transformations two and one. The current sources must be equal in transformation three or else Kirchhoff's law would not be valid at the node indicated, and the circuit would be a contradiction.

Source transformation five shows that the circuit M_1 can be removed without altering any of the voltages and currents inside N. Whether M_1 is connected as shown or is removed, the voltage applied to N remains V_s. However, the current supplied by the source V_s changes from I_s to I_1.

Source transformation six shows that circuit M_2 can be replaced by a short circuit without affecting voltages and currents in N. Whether M_2 is in series with the current source I_1 as shown or removed, the current applied to N is the same. However, if network M_2 is removed, then the voltage across the current source changes from V_s to V_1.

Source transformation seven is sometimes termed a Thévenin circuit to Norton circuit transformation. This transformation, as shown by the double-headed arrow, can be used in the reverse direction. Thévenin's theorem is discussed thoroughly later in this section.

Source transformation eight is sometimes described as "pushing a voltage source through a node," but we will term it as "splitting a voltage source." Loop equations remain the same with this transformation, and the current leaving network N through the lowest wire continues to be I_s.

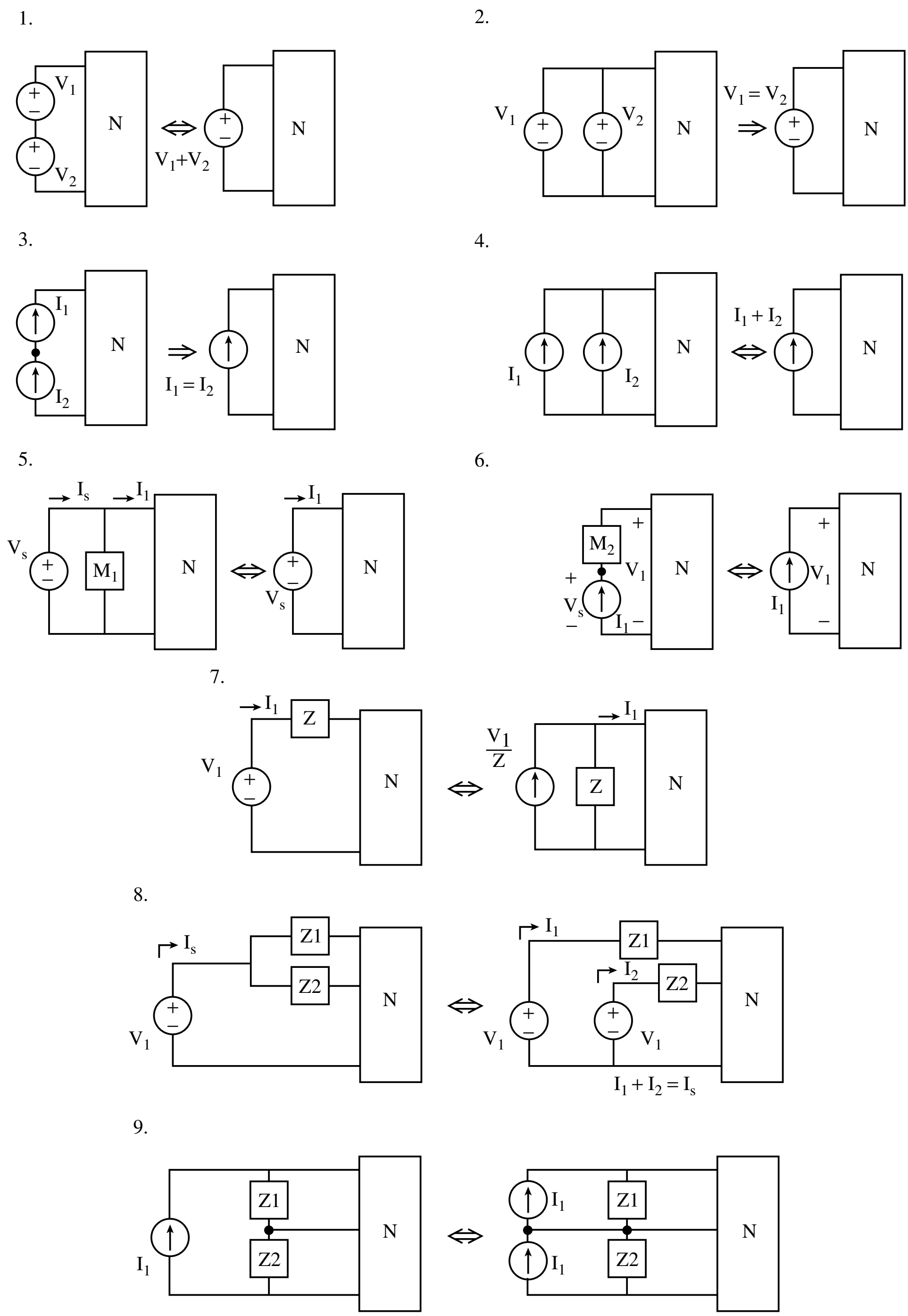
Source transformation nine shows that if a current source is not in parallel with one element, then it can be "split" as shown. Now, each one of the current sources I_1 has an impedance in parallel. Thus, analysis of network N may be simplified because source transformation seven can be applied.

Source transformations cannot be applied to all networks, but when they can be employed, they usually yield useful simplifications of the network.

Example 2. Use source transformations to find V_0 for the network shown. Initial current through the inductor in the network is zero.

Solution. The network can be readily simplified by employing source transformation five from Table 7.4 to eliminate R_1 and also I_2. Then, source transformation six can be used to eliminate V_1 because it is an element in series with a current source. The results to this point are illustrated in Figure 7.13(a). If we

TABLE 7.4 Source Transformations



N is an arbitrary network in which analysis for a voltage or current is to be performed. M_1 is an arbitrary one-port network or network element except a voltage source. M_2 is an arbitrary one-port network or network element except a current source. It is assumed there is no magnetic coupling between *N* and M_1 or M_2. There are no dependent sources in *N* in Source Transformation 5 that depend on I_s. Furthermore, there are no dependent sources in *N* in Source Transformation 6 that depend on V_s. However, M_1 and M_2 can have dependent sources that depend on voltages or currents in *N*. *Z*, Z_1 and Z_2 are one-port impedances.

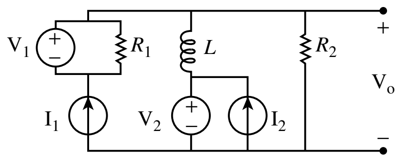


FIGURE 7.12 Network for Example 2.

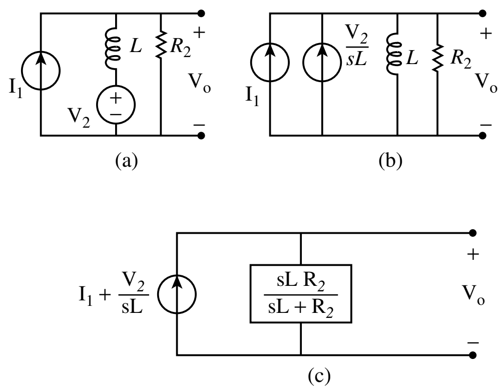


FIGURE 7.13 (a, b, c) Applications of source transformations to the network in Figure 7.12.

then apply transformation seven, we obtain the network in Figure 7.13(b). Next, we can apply transformation four to obtain the single loop network in Figure 7.13(c). The output voltage can be written in the frequency domain as

$$V_0 = \left(I_1 + \frac{V_2}{sL}\right)\left(\frac{sLR_2}{sL + R_2}\right)$$

Source transformations can often be used advantageously with the following theorems. □

Dividers

Current dividers and voltage dividers are circuits that are employed frequently, especially in the design of electronic circuits. Thus, dividers must be analyzed quickly. The relationships derived next satisfy this need.

Figure 7.14 is a current divider circuit. The source current I_s divides between the two impedances, and we wish to determine the current through Z_2. Writing a loop equation for the loop indicated, we have

$$I_2 Z_2 - (I_s - I_2) Z_1 = 0 \tag{7.26}$$

from which we obtain

$$I_2 = I_s \frac{Z_1}{Z_1 + Z_2} \tag{7.27}$$

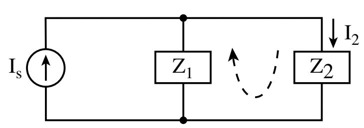


FIGURE 7.14 A current divider.

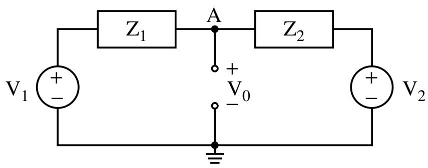


FIGURE 7.15 Enhanced voltage divider.

A circuit that we term an **enhanced** voltage divider is depicted in Figure 7.15. This circuit contains two voltage sources instead of the usual single source, but the enhanced voltage divider occurs more often in electronic circuits. Writing a node equation at node A and solving for V_0, we obtain

$$V_0 = \frac{V_1 Z_2 + V_2 Z_1}{Z_1 + Z_2} \tag{7.28}$$

If V_2, for example, is zero, then the results from the enhanced voltage divider reduce to those of the single source voltage divider.

Example 3. Use (7.28) to find V_0 for the network in Figure 7.16.

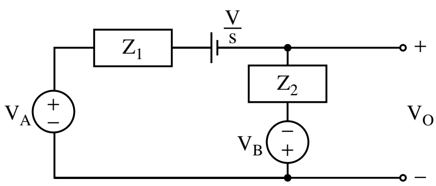


FIGURE 7.16 Circuit for Example 3.

Solution. The network in Figure 7.16 matches with the network used to derive (7.28) even though it is drawn somewhat differently and has three voltage sources instead of two. However, we can use (7.28) to write the answer for V_0 by inspection.

$$V_0 = \frac{\left(V_A - (V/s)\right) Z_z - V_B Z_1}{Z_1 + Z_2}$$

□

The following example illustrates the use of source transformations together with the voltage divider.

Example 4. Find V_0 for the network shown in Figure 7.17. The units of K, the coefficient of the dependent source, are ohms, and the capacitor is initially uncharged.

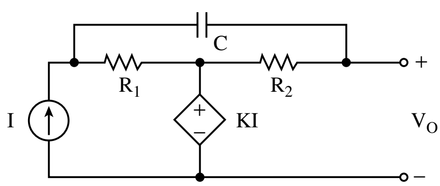


FIGURE 7.17 Network for Example 4.

Solution. We note that the dependent voltage source is not in series with any one particular element and that the independent current source is not in parallel with any one particular element. However, we can split both the voltage source and the current source using source transformations eight and nine, respectively, from Table 7.4. Then, employing transformations five and seven, we obtain the network configuration depicted in Figure 7.18, for which we can use the voltage divider to write:

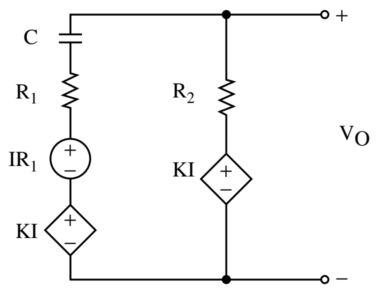


FIGURE 7.18 Results after employing source transformations on the network in Figure 7.17.

$$V_0 = \frac{I(K+R_1)R_2 + KI(R_1 + (1/sC))}{R_1 + R_2 + (1/sC)}$$

It should be mentioned that the method used to find V_0 in this example is not the most efficient one. For example, loops can be chosen for the network in Figure 7.17, so only one unknown loop is current. However, source transformations and dividers become more powerful analysis tools as they are coupled with additional network theorems.

Superposition

Superposition is a property of all linear networks, and whether it is used directly in the analysis of a network or not, it is a concept that is valuable in thinking about LLFT networks. Consider the LLFT network shown in Figure 7.19 in which, say, we wish to solve for I_1. Assume the network has n independent loops, and, for simplicity, assume no sources are within the box in the figure and that initial voltages across capacitors and initial currents through inductors are zero or are represented by independent sources external to the box. Note that one dependent source is shown in Figure 7.19 that depends on a voltage V_x in the network and that two independent sources, V_1 and V_2, are applied to the network. If loops are chosen so that each source has only one loop current flowing through it as indicated in Figure 7.19, then the loop equations can be written as

$$\begin{bmatrix} V_1 \\ V_2 \\ KV_x \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \tag{7.29}$$

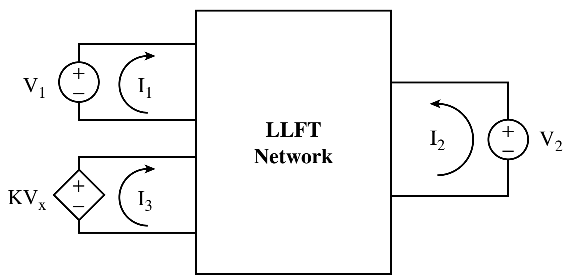


FIGURE 7.19 LLFT network with three voltage sources, of which one is dependent.

where the elements of the loop impedance matrix are defined in the section describing network functions. Solving for I_1 using Cramer's rule, we have:

$$I_1 = V_1\frac{\Delta_{11}}{\Delta} + V_2\frac{\Delta_{21}}{\Delta} + KV_x\frac{\Delta_{31}}{\Delta} \tag{7.30}$$

where Δ is the determinant of the loop impedance matrix, and Δ_{j1}, $j = 1, 2, 3$, are cofactors. The expression for I_1 given in (7.30) is an intermediate and not a finished solution. The finished solution would express I_1 in terms of the independent sources and the parameters (Rs, Ls, Cs, Ms, and Ks) of the network and not in terms of an unknown V_x. Thus, one normally has to eliminate V_x from the expression for I_1; but the intermediate expression for I_1 illustrates superposition. There are three components that add up to I_1 in (7.30) — one for each source including one for the dependent source. Furthermore, we see that each source is multiplied by a transadmittance (or a driving-point admittance in the case of V_1). Thus, we can write:

$$I_1 = V_1Y_{11} + V_2Y_{12} + KV_xY_{13} \tag{7.31}$$

where each admittance is found from the port at which a voltage source (whether independent or dependent) is applied. The response variable for each of these admittances is I_1 at port 1.

The simple derivation that led to (7.30) is easily extended to both types of independent excitations (voltage sources and current sources) and to all four types of dependent sources. The generalization of (7.30) leads to the conclusion:

> To apply superposition in the analysis of a network containing at least one independent source and a variety of other sources, dependent or independent, one finds the contribution to the response from each source in turn with all other source, dependent or independent, properly removed and then adds the individual contributions to obtain the total response. No distinction is made between independent and dependent sources in the application of superposition other than requiring the network to have at least one independent source.

However, if dependent sources are present in the network, the quantities (call them V_x and I_x) on which the dependent sources depend must often be eliminated from the answer by additional analysis if the answer is to be useful unless V_x or I_x are themselves the variables of independent sources or the quantities sought in the analysis.

Some examples will illustrate the procedure.

Example 5. Find V_0 for the circuit shown using superposition. In this circuit, only independent sources are present.

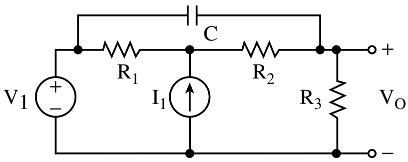


FIGURE 7.20 Network for Example 5.

Solution. Two sources in the network, therefore, we abstract two fictitious networks from Figure 7.20. The first is shown in Figure 7.21(a) and is obtained by properly removing the current source I_1 from the original network. The impedance of the capacitor can then be combined in parallel with $R_1 + R_2$, and the contribution to V_0 from V_1 can be found using a voltage divider. The result is

$$V_0 \text{ due to } V_1 = V_1\frac{s + \dfrac{1}{C(R_1+R_2)}}{s + \dfrac{R_1+R_2+R_3}{C(R_1+R_2)R_3}}$$

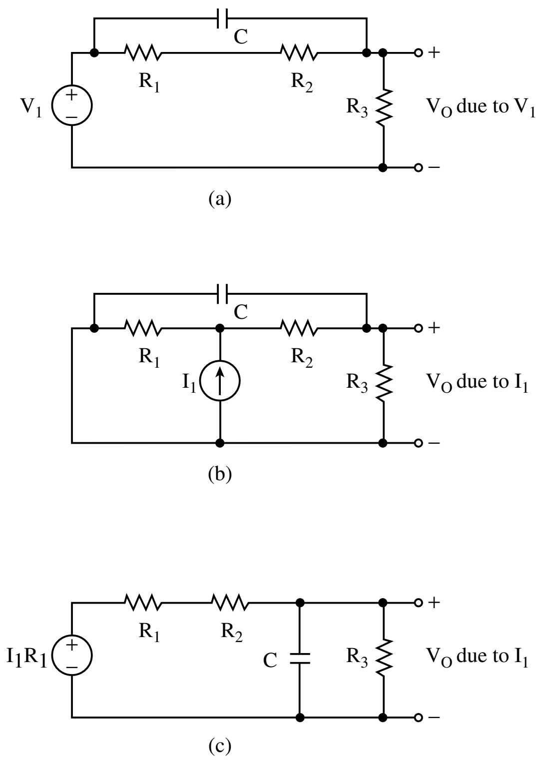


FIGURE 7.21 (a, b, c) Steps in the use of superposition for finding the response to two independent sources.

The second fictitious network, shown in Figure 7.21(b), is obtained form the original network by properly removing the voltage source V_1. Redrawing the circuit and employing source transformation seven (in reverse) yields the circuit in Figure 7.21(c). Again, employing a voltage divider, we have

$$V_0 \text{ due to } I_1 = I_1 \frac{\dfrac{R_1}{C(R_1 + R_2)}}{s + \dfrac{R_1 + R_2 + R_3}{C(R_1 + R_2)R_3}}$$

Then, adding the two contributions, we obtain

$$V_0 = \frac{V_1\left[s + \dfrac{1}{C(R_1 + R_2)}\right] + I_1 \dfrac{R_1}{C(R_1 + R_2)}}{s + \dfrac{R_1 + R_2 + R_3}{C(R_1 + R_2)R_3}}$$

The next example includes a dependent source.

Example 6. Find i in the network shown in Figure 7.22 using superposition.

Solution. Since there are two sources, we abstract two fictitious networks from Figure 7.22. The first one is shown in Figure 7.23(a) and is obtained by properly removing the dependent current source. Thus,

$$i \text{ due to } v_1 = \frac{v_1}{R_1 + R_2}$$

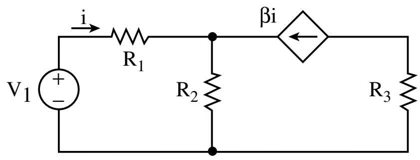


FIGURE 7.22 Network for Example 6.

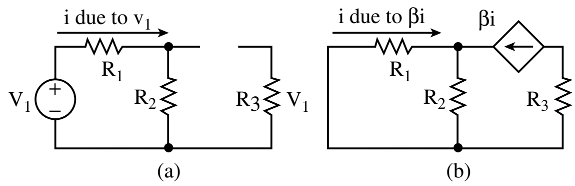


FIGURE 7.23 (a, b) Steps in the application of superposition to the network in Figure 7.22.

Next, voltage source v_1 is properly removed yielding the fictitious network in Figure 7.23(b). An important question immediately arises about this network. Namely, why is not i in this network zero? The reason i is not zero is that the network in Figure 7.23(b) is merely an abstracted network that concerns a step in the analysis of the original circuit. It is an artifice in the application of superposition, and the dependent source is considered to be independent for this step. Thus,

$$i \text{ due to } \beta i = -\frac{\beta i R_2}{R_1 + R_2}$$

Adding the two contributions, we obtain the intermediate result:

$$i = \frac{v_1}{R_1 + R_2} - \frac{\beta i R_2}{R_1 + R_2}$$

Collecting the terms containing i, we obtain the finished solution for i:

$$i = \frac{v_1}{(\beta + 1)R_2 + R_1}$$

We note that the finished solution depends only on the independent source v_1 and parameters of the network, which are R_1, R_2, and β. □

The following example involves a network in which a dependent source depends on a voltage that is neither the voltage of an independent source nor the voltage being sought in the analysis.

Example 7. Find V_0 using superposition for the network shown in Figure 7.24. Note that K, the coefficient of the VCCS, has the units of siemens.

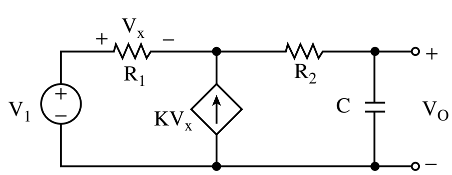


FIGURE 7.24 Network for Example 7.

Solution. When the dependent current source is properly removed, the network reduces to a simple voltage divider, and the contribution to V_0 due to V_1 can be written as:

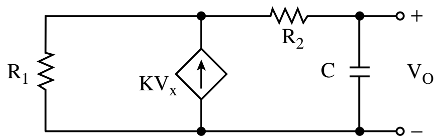


FIGURE 7.25 Fictitious network obtained when the voltage source is properly removed in Figure 7.24.

$$V_0 \text{ due to } V_1 = V_1 \frac{1}{sC(R_1 + R_2) + 1}$$

Then, reinserting the current source and properly removing the voltage source, we obtain the fictitious network shown in Figure 7.25. Using the current divider to obtain the current flowing through the capacitor and then multiplying this current by the impedance of the capacitor, we have:

$$V_0 \text{ due to } KV_x = \frac{KV_x R_1}{sC(R_1 + R_2) + 1}$$

Adding the individual contributions to form V_0 provides the equation

$$V_0 = \frac{V_1 + KV_x R_1}{sC(R_1 + R_2) + 1}$$

This is a valid expression for V_0. It is not a finished expression however, because it includes V_x, an unknown voltage. Superposition has taken us to this point in the analysis, but more work must be done to eliminate V_x. However, superposition can be applied again to solve for V_x, or other analysis tools can be used. The results for V_x are:

$$V_x = \frac{V_1 sCR_1}{sC[R_1 + R_2 + R_1 R_2 K] + R_1 K + 1}$$

Then, eliminating V_x from the equation for V_0, we obtain the finished solution as:

$$V_0 = V_1 \frac{R_1 K + 1}{sC[R_1 + R_2 + R_1 R_2 K] + R_1 K + 1}$$

Clearly, superposition is not the most efficient technique to use to analyze the network in Figure 7.24. Analysis based on a node equation written at the top end of the current source would yield a finished result for V_0 with less algebra. However, this example does illustrate the application of superposition when a dependent source depends on a rather arbitrary voltage in the network. □

If the dependent current source in the previous example depended on V_1 instead of on the voltage across R_1, the network would be a different network. This is illustrated by the next example.

Example 8. Use superposition to determine V_0 for the circuit in Figure 7.26.

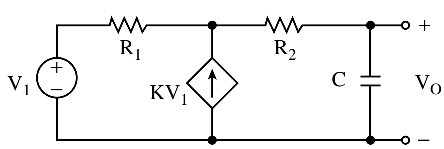


FIGURE 7.26 Network for Example 8.

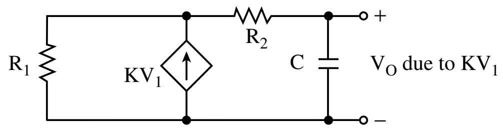


FIGURE 7.27 A step in the application of superposition to the network in Figure 7.26.

Solution. If the current source is properly removed, the results are the same as for the previous example. Thus,

$$V_0 \text{ due to } V_1 = \frac{V_1}{sC(R_1 + R_2) + 1}$$

Then, if the current source is reinserted, and the voltage source is properly removed, we have the circuit depicted in Figure 7.27. A question that can be asked for this circuit is why include the dependent source KV_1 if the voltage on which it depends, namely V_1, has been set to zero? However, the network shown in Figure 7.27 is merely a fictitious network that serves as an aid in the application of superposition, and superposition deals with all sources, whether they are dependent or independent, as if they were independent. Thus, we can write:

$$V_0 \text{ due to } KV_1 = \frac{KV_1R_1}{sC(R_1 + R_2) + 1}$$

Adding the contributions to form V_0, we obtain

$$V_0 = V_1 \frac{KR_1 + 1}{sC(R_1 + R_2) + 1}$$

and this is the finished solution.

In this example, we did not have the task of eliminating an unknown quantity from an intermediate result for V_0 because the dependent source depended on an independent source V_1, which is assumed to be known. □

Superposition is often useful in the analysis of circuits having only independent sources, but it is especially useful in the analysis of some circuits having both independent and dependent sources because it deals with all sources as if they were independent.

Thévenin's Theorem

Thévenin's theorem is useful in reducing the complexity of a network so that analysis of the network for a particular voltage or current can be performed more easily. For example, consider Figure 7.28(a), which is composed of two subnetworks A and B that have only two nodes in common. In order to facilitate analysis in subnetwork B, it is convenient to reduce subnetwork A to the network in Figure 7.28(b) which is termed the Thévenin equivalent of subnetwork A. The requirement on the Thévenin's equivalent

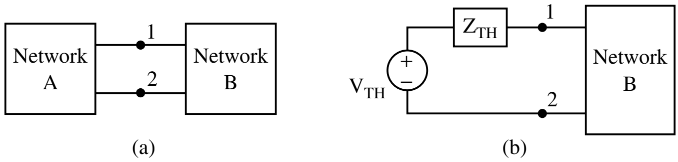


FIGURE 7.28 (a) Two subnetworks having a common pair of terminals. (b) The Thévenin equivalent for subnetwork A.

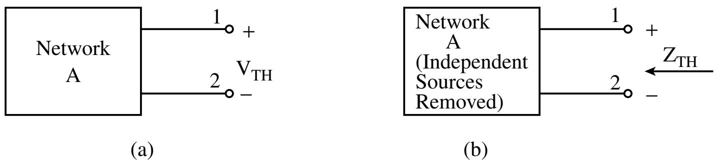


FIGURE 7.29 (a) Network used for finding V_{TH}. (b) Network used for obtaining Z_{TH}.

network is that, when it replaces subnetwork A, the voltages and currents in subnetwork B remain unchanged. We assume that no inductive coupling occurs between the subnetworks, and that dependent sources in B are not dependent on voltages or currents in A. We also assume that subnetwork A is an LLFT network, but subnetwork B does not have to meet this assumption.

To find the Thévenin equivalent network, we need only determine V_{TH} and Z_{TH}. V_{TH} is found by unhooking B from A and finding the voltage that appears across the terminals of A. In other words, we abstract a fictitious network from the complete network as depicted in Figure 7.29(a), and find the voltage that appears between the terminals that were common to B. This voltage is V_{TH}.

Z_{TH} is also obtained from a fictitious network that is created from the fictitious network used for finding V_{TH} by properly removing all independent sources. The effects that dependent sources have on the procedure are discussed later in this section. The fictitious network used for finding Z_{TH} is depicted in Figure 7.29(b). Oftentimes, the expression for Z_{TH} cannot be found by mere inspection of this network, and, therefore, we must excite the network in Figure 7.29(b) by a voltage source or a current source and find an expression for the other variable at the port in order to find Z_{TH}.

Example 9. Find the Thévenin equivalent of subnetwork A in Figure 7.30.

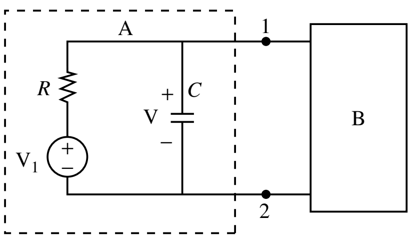


FIGURE 7.30 Network for Example 9.

Solution. No dependent sources exist in subnetwork A, but the capacitor has an initial voltage V across it. However, the charged capacitor can be represented by an uncharged capacitor in series with a transformed voltage source V/s. The fictitious network used for finding V_{TH} is given in Figure 7.31(a).

It should be noted that although subnetwork B has been removed and the two terminals that were connected to B are now "open circuited" in Figure 7.31(a), current is still flowing in network A. V_{TH} is easily obtained using a voltage divider:

$$V_{TH} = \frac{V_1(s) + VCR}{sCR + 1}$$

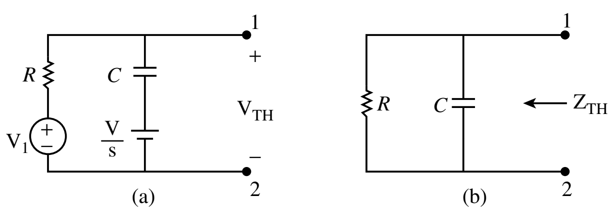


FIGURE 7.31 (a) Network of finding V_{TH}. (b) Network that yields Z_{TH}.

Z_{TH} is obtained from the fictitious network in Figure 7.31(b), which is obtained by properly removing the independent source and the voltage representing the initial voltage across the capacitor in Figure 7.31(a). We see by inspection that $Z_{TH} = R/(sCR + 1)$. Thus, if subnetwork A is removed from Figure 7.30 and replaced by the Thévenin equivalent network, the voltages and currents in subnetwork B remain unchanged.

It is assumed that B in Figure 7.28 has no dependent sources that depend on voltages or currents in A, although dependent sources in B can depend on voltages and currents in B. However, A can have dependent sources, and these dependent sources create a modification in the procedure for finding the Thévenin equivalent network. There may be dependent sources in A that depend on voltages and currents that also exist in A. We call these dependent sources Case I-dependent sources. There may also be dependent sources in A that depend on voltages and currents in B, and we label these sources as Case II-dependent sources. Then, the procedure for finding the Thévenin equivalent network is:

> V_{TH} is the voltage across the terminals of Figure 7.29(a). The voltages and currents that Case I-dependent sources depend on must be eliminated from the expression for V_{TH} unless they happen to be the voltages of independent voltage sources or the currents of independent current sources in A. Otherwise, the expression for V_{TH} would not be a finished solution. However, Case II-dependent sources are handled as if they were independent sources. That is, Case II-dependent sources are included in the results for V_{TH} just as independent sources would be.
>
> Z_{TH} is the impedance looking into the terminals in Figure 7.29(b). In this fictitious network, independent sources are properly removed and *Case II-dependent sources are properly removed.* Case I-dependent sources remain in the network and influence the result for the Thévenin impedance. The finished solution for Z_{TH} is a function only of the parameters of the network in Figure 7.29(b) which are Rs, Ls, Cs, Ms (there may be inductive coupling between coils in this network), and the coefficients of the Case I-dependent sources.

Thus, Case II-dependent sources, sources that depend on voltages or currents in subnetwork B, are uniformly treated as if they were independent sources in finding the Thévenin equivalent network. Some examples will clarify the issue.

Example 10. Find the Thévenin equivalent network for subnetwork A in Figure 7.32. Assume the initial current through the inductor is zero.

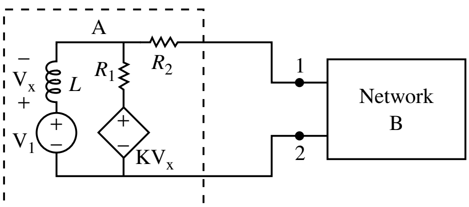


FIGURE 7.32 Network for Example 10.

Solution. There is one independent source and one Case I-dependent source. Figure 7.33(a) depicts the fictitious network to be analyzed to obtain V_{TH}. No current is flowing through R_2 in this figure, therefore, we can write $V_{TH} = V_1 - V_x$. To eliminate V_x from our intermediate expression for V_{TH}, we can use the results of the enhanced voltage divider to write:

$$V_{TH} = \frac{V_1 R_1 + sKL(V_1 - V_{TH})}{R_1 + sL}$$

The finished solution for V_{TH} is

$$V_{TH} = V_1 \frac{sKL + R_1}{(K+1)sL + R_1}$$

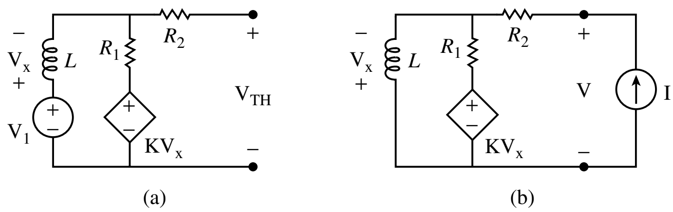


FIGURE 7.33 (a) Abstracted network for finding V_{TH}. (b) Abstracted network for finding Z_{TH}.

Z_{TH} is obtained from Figure 7.33(b) where a current source excitation is shown already applied to the fictitious network. Two node equations, with unknown node voltages V and $-V_x$, enable us to obtain I in terms of V while eliminating V_x. We also note that Z_{TH} consists of resistor R_2 in series with some unknown impedance, so we could remove R_2 (replaced it by a short) if we remember to add it back later. The finished result for Z_{TH} is

$$Z_{TH} = R_2 + \frac{sLR_1}{(K+1)sL + R_1}$$

The following example involves a network having a Case II-dependent source. □

Example 11. Find the Thévenin equivalent network for subnetwork A in the network illustrated in Figure 7.34. In this instance, subnetwork B is outlined explicitly.

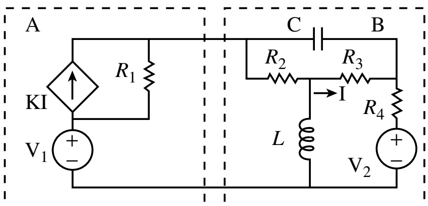


FIGURE 7.34 Network for Example 11.

Solution. Subnetwork A contains one independent source and one Case II-dependent source. Figure 7.35(a) is the abstracted network for finding V_{TH}. Thus,

$$V_{TH} = V_1 + KIR_1$$

Then, both V_1 and the dependent source KI are deleted from Figure 7.35(a) to obtain Figure 7.35(b), the network used for finding Z_{TH}. Thus, $Z_{TH} = R_1$.

Of course, the subnetwork for which the Thévenin equivalent is being determined may have both Case I- and Case II-dependent sources, but these sources can be handled concurrently using the procedures given previously.

Special conditions can arise in the application of Thévenin's theorem. One condition is $Z_{TH} = 0$ and the other is $V_{TH} = 0$. The conditions for which Z_{TH} is zero are:

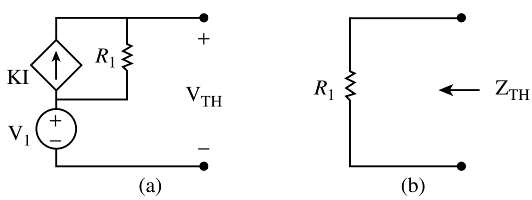


FIGURE 7.35 (a) Network used to find V_{TH}. (b) Network for finding Z_{TH}.

1. If the circuit (subnetwork A) for which the Thévenin equivalent is being determined has an independent voltage source connected between terminals 1 and 2, then $Z_{TH} = 0$. Figure 7.36(a) illustrates this case.
2. If subnetwork A has a *dependent* voltage source connected between terminals 1 and 2, then Z_{TH} is zero *provided* neither of the port variables associated with the port formed by terminals 1 and 2 is coupled back into the network. Figure 7.36(b) is a subnetwork A for which Z_{TH} is zero. However, Figure 7.36(c) depicts a subnetwork A in which the port variable I is coupled back into A by the dependent source K_1I. If I is considered to be a variable of subnetwork A so that K_1I is a Case I-dependent source, then Z_{TH} is not zero.

The other special condition, $V_{TH} = 0$, occurs if subnetwork A contains only Case I-dependent sources, no independent sources, and no Case II-dependent sources. An example of such a network is given in Figure 7.36(d). With subnetwork B disconnected, subnetwork A is a dead network, and its Thévenin voltage is zero.

The network in Figure 7.36(c) is of interest because the dependent source K_1I can be considered as a Case I- or a Case II-dependent source hinging on whether I is considered a variable of subnetwork A or B.

Example 12. Solve for I in Figure 7.36(c) using two versions of the Thévenin equivalent for subnetwork A. For the first version, consider I to be associated with A, and therefore both dependent sources are Case I-dependent sources. In the second version, consider I to be associated with B.

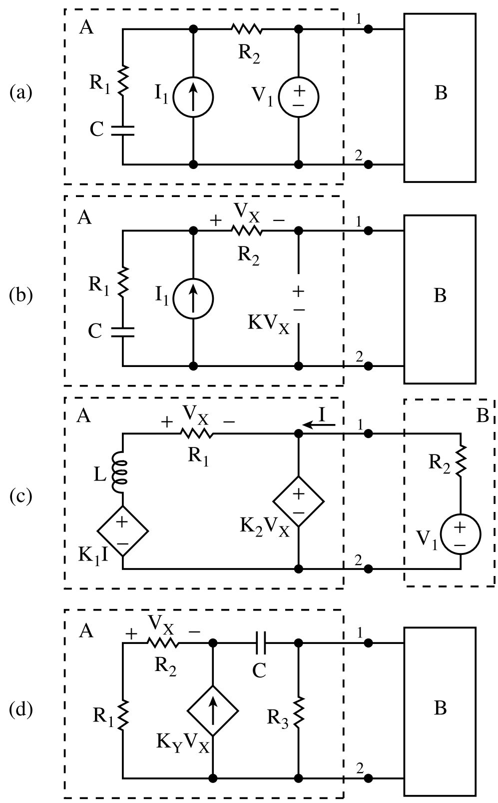


FIGURE 7.36 Special cases of Thévenin's theorem. (a, b) Z_{TH} equals zero. (c) A port variable is coupled back into A. (d) V_{TH} is zero.

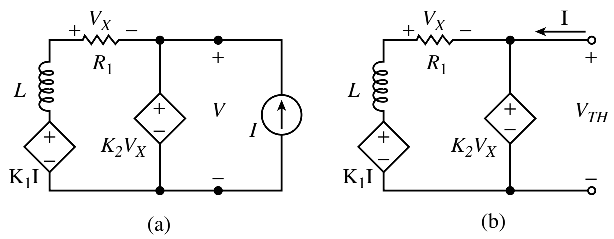


FIGURE 7.37 (a) Network for finding Z_{TH} when both sources are Case I-dependent sources. (b) Network for finding V_{TH} when a Case II-dependent source exists in the network.

Solution. If I is considered as associated with A, then V_{TH} is zero by inspection because A contains only Case I-dependent sources. Figure 7.37(a) depicts subnetwork A with a current excitation applied in order to determine Z_{TH}. Clearly, $V = K_2V_x$. Also, writing a loop equation in the loop encompassed by the two dependent sources, we obtain

$$K_1I - \frac{V_x}{R_1}(sL + R_1) = V$$

Eliminating V_x, we have

$$\frac{V}{I} = Z_{TH} = \frac{K_1K_2R_1}{sL + R_1(K_2+1)}$$

Once Z_{TH} is obtained, it is an easy matter to write from Figure 7.36(c):

$$I = \frac{V_1}{R_2 + Z_{TH}} = \frac{V_1[sL + R_1(K_2+1)]}{sLR_2 + R_1R_2(K_2+1) + K_1K_2R_1}$$

If I is associated with B, then V_{TH} is found from the network in Figure 7.37(b) with the source K_1I treated as if it were independent. The equation for V_{TH} may contain the variable I, but V_x must be eliminated from the finished expression for V_{TH}. We obtain

$$V_{TH} = I\frac{K_1K_2R_1}{sL + R_1(K_2+1)}$$

Also, Z_{TH} is zero because if K_1I is removed, subnetwork A reduces to a network with only a Case I-dependent source and a port variable is not coupled back into the network. Finally, I can be written as:

$$I = \frac{V_1 - V_{TH}}{R_2}$$

which yields the same result for I as was found previously. □

The following example illustrates the interplay that can be achieved among these theorems and source transformations.

Example 13. Find V_0/V_1 for the bridged T network shown in Figure 7.38.

Solution. The application of source transformation eight to the network yields the ladder network in Figure 7.39(a). Thévenin's theorem is particularly useful in analyzing ladder networks. If it is applied to

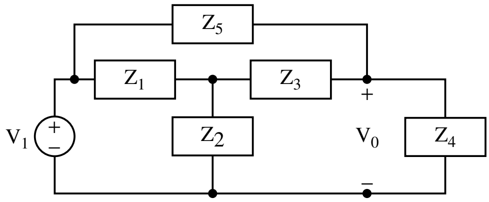


FIGURE 7.38 Network for Example 13.

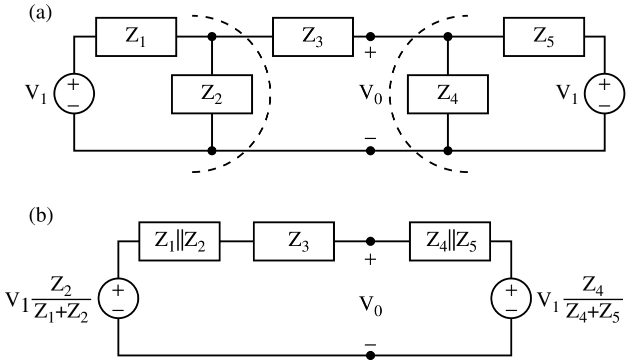


FIGURE 7.39 (a) Results after applying source transformation eight to the network shown in Figure 7.38. (b) Results of two applications of Thévenin's theorem.

the left and right sides of the network, taking care not to obscure the nodes between which V_0 exists, we obtain the single loop network in Figure 7.39(b). Then, using a voltage divider, we obtain

$$\frac{V_0}{V_1} = \frac{Z_4\left[Z_1(Z_2+Z_3)+Z_2Z_3+Z_2Z_5\right]}{(Z_1+Z_2)(Z_3Z_4+Z_3Z_5+Z_4Z_5)+Z_1Z_2(Z_4+Z_5)}$$

Norton's Theorem

If a source transformation is applied to the Thévenin equivalent network consisting of V_{TH} and Z_{TH} in Figure 7.28(b), then a Norton equivalent network, illustrated in Figure 7.40(a), is obtained. The current source $I_{sc} = V_{TH}/Z_{TH}$, $Z_{TH} \neq 0$. If Z_{TH} equals zero in Figure 7.28(b), then the Norton equivalent network does not exist. The subscripts "*sc*" on the current source stand for short circuit and indicate a procedure for finding the value of this current source. To find I_{sc} for subnetwork A in Figure 7.28(a), we disconnet subnetwork B and place a short circuit between nodes 1 and 2 of subnetwork A. Then, I_{sc} is the current flowing through the short circuit in the direction indicated in Figure 7.40(b). I_{sc} is zero if subnetwork A has only Case I-dependent sources and no other sources. Z_{TH} is found in the same manner as for Thévenin's theorem.

It is sometimes more convenient to find I_{sc} and V_{TH} instead of Z_{TH}.

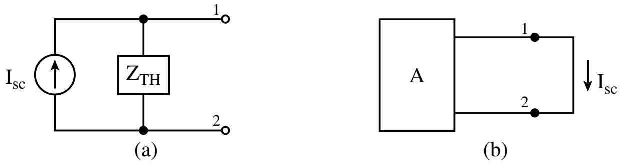


FIGURE 7.40 (a) Norton equivalent network. (b) Reference direction for I_{sc}.

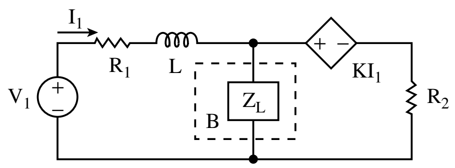


FIGURE 7.41 Network for Example 14.

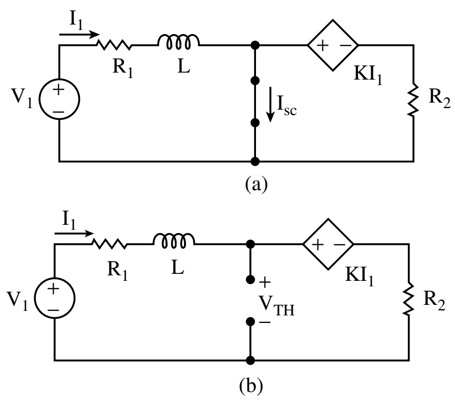


FIGURE 7.42 (a) Network for finding $I_{sc.}$ (b) Network for V_{TH}.

Example 14. Find the Norton equivalent for the network "seen" by Z_L in Figure 7.41. That is, Z_L is subnetwork B and the rest of the network is A, and we wish to find the Norton equivalent network for A.

Solution. Figure 7.42(a) is the network with Z_L replaced by a short circuit. An equation for I_{sc} can be obtained quickly using superposition. This yields

$$I_{sc} = I_1 + \frac{KI_1}{R_2}$$

but I_1 must be eliminated from this equation. I_1 is obtained as: $I_1 = V_1/(sL + R_1)$. Thus,

$$I_{sc} = \frac{V_1\left(1 + \frac{K}{R_2}\right)}{sL + R_1}$$

V_{TH} is found from the network shown in Figure 7.42(b). The results are:

$$V_{TH} = \frac{V_1(K + R_2)}{sL + R_1 + R_2 + K}$$

Z_{TH} can be found as V_{TH}/I_{sc}.

Thévenin's and Norton's Theorems and Network Equations

Thévenin's and Norton's theorems can be related to loop and node equations. Here, we examine the relationship to loop equations by means of the LLFT network in Figure 7.43. Assume that the network N in Figure 7.43 has n independent loops with all the loop currents chosen in the same direction. Without loss of generality, assume that only one loop current, say I_1, flows through Z_L as shown so that Z_L appears

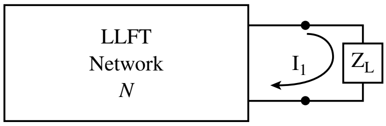


FIGURE 7.43 An LLFT network N with n independent loops.

in only one loop equation. For simplicity, assume that no dependent sources or inductive couplings in N exist, and that all current sources have been source-transformed so that only voltage source excitations remain. Then the loop equations are

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \tag{7.32}$$

where V_i, $i = 1, 2, \ldots, n$, is the sum of all voltage sources in the *ith* loop. Thus, V_i may consist of several terms, some of which may be negative depending on whether a voltage source is a voltage rise or a voltage drop. Also, the impedances z_{ij} are given by

$$z_{ij} = \pm\left[R_{ij} + sL_{ij} + \frac{D_{ij}}{s}\right] \tag{7.33}$$

where $i, j = 1, 2, \ldots, n$, and where the plus sign is taken if $i = j$, and the minus sign is used if $i \neq j$. R_{ij} is the sum of the resistances in loop i if $i = j$, and R_{ij} is the sum of the resistances common to loops i and j if $i \neq j$. Similar statements apply to the inductances L_{ij} and to the reciprocal capacitances D_{ij}. The currents I_i, $i = 1, 2, \ldots, n$, are the unknown loop currents.

Note that Z_L is included only in z_{11} so that z_{11} can be written as $z_{11} = z_{11A} + Z_L$, where z_{11A} is the sum of all the impedances around loop one except Z_L. Solving for I_1 using Cramer's rule, we have

$$I_1 = \frac{\begin{vmatrix} V_1 & z_{12} & \cdots & z_{1n} \\ V_2 & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_n & z_{n2} & \cdots & z_{nn} \end{vmatrix}}{\Delta} \tag{7.34}$$

where Δ is the determinant of the loop impedance matrix. Thus, we can write

$$I_1 = \frac{V_1\Delta_{11} + V_2\Delta_{21} + \cdots + V_n\Delta_{n1}}{z_{11}\Delta_{11} + z_{21}\Delta_{21} + \cdots + z_{n1}\Delta_{n1}} \tag{7.35}$$

or

$$I_1 = \frac{V_1 + V_2\dfrac{\Delta_{21}}{\Delta_{11}} + \cdots + V_n\dfrac{\Delta_{n1}}{\Delta_{11}}}{z_{11} + z_{21}\dfrac{\Delta_{21}}{\Delta_{11}} + \cdots + z_{n1}\dfrac{\Delta_{n1}}{\Delta_{11}}} \tag{7.36}$$

where Δ_{ij} are cofactors of the loop impedance matrix. Then, forming the product of I_1 and Z_L, we have:

$$I_1 Z_L = \frac{Z_L\left(V_1 + V_2\frac{\Delta_{21}}{\Delta_{11}} + \cdots + V_n\frac{\Delta_{n1}}{\Delta_{11}}\right)}{Z_L + z_{11A} + z_{21}\frac{\Delta_{21}}{\Delta_{11}} + \cdots + z_{n1}\frac{\Delta_{n1}}{\Delta_{11}}} \tag{7.37}$$

If we take the limit of $I_1 Z_L$ as Z_L approaches infinity, we obtain the "open circuit" voltage V_{TH}. That is,

$$\lim_{Z_L\to\infty} I_1 Z_L = V_{TH} = \left(V_1 + V_2\frac{\Delta_{21}}{\Delta_{11}} + \cdots + V_n\frac{\Delta_{n1}}{\Delta_{11}}\right) \tag{7.38}$$

and if we take the limit of I_1 as Z_L approaches zero, we obtain the "short circuit" current I_{sc}:

$$\lim_{Z_L\to 0} I_1 = I_{sc} = \frac{V_{TH}}{z_{11A} + z_{21}\frac{\Delta_{21}}{\Delta_{11}} + \cdots + z_{n1}\frac{\Delta_{n1}}{\Delta_{11}}} \tag{7.39}$$

Finally, the quotient of V_{TH} and I_{sc} yields:

$$\frac{V_{TH}}{I_{sc}} = Z_{TH} = z_{11A} + z_{21}\frac{\Delta_{21}}{\Delta_{11}} + \cdots + z_{n1}\frac{\Delta_{n1}}{\Delta_{11}} \tag{7.40}$$

If network N contains coupled inductors (but not coupled to Z_L), then some elements of the loop impedance matrix may be modified in value and sign. If network N contains dependent sources, then auxiliary equations can be written to express the quantities on which the dependent sources depend in terms of the independent excitations and/or the unknown loop currents. Thus, dependent sources may modify the elements of the loop impedance matrix in value and sign, and they may modify the elements of the excitation column matrix $[V_i]$. Nevertheless, we can obtain expressions similar to those obtained previously for V_{TH} and I_{sc}. Of course, we must exclude from this illustration dependent sources that depend on the voltage across Z_L because they violate the assumption that Z_L appears in only one loop equation and are beyond the scope of this discussion.

The π-T Conversion

The π-T conversion is employed for the simplification of circuits, especially in power systems analysis. The "π" refers to a circuit having the topology shown in Figure 7.44. In this figure, the left-most and right-most loop currents have been chosen to coincide with the port currents for convenience of notation only.

A circuit having the topology shown in Figure 7.45 is referred to as a "T" or as a "Y." We wish to determine equations for Z_1, Z_2, and Z_3 in terms of Z_A, Z_B, and Z_C so that the π can be replaced by a T

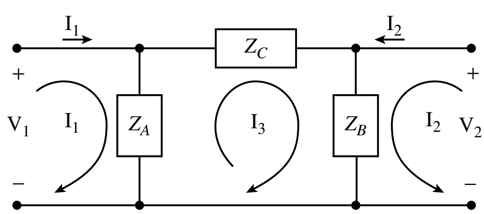


FIGURE 7.44 A π network shown with loop currents.

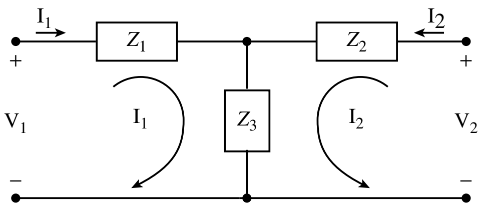


FIGURE 7.45 A T network.

without affecting any of the port variables. In other words, if an overall circuit contains a π subcircuit, we wish to replace the π subscript with a T subscript without disturbing any of the other voltages and currents within the overall circuit. To determine what Z_1, Z_2, and Z_3 should be, we first write loop equations for the π network. The results are:

$$V_1 = I_1 Z_A - I_3 Z_A \tag{7.41}$$

$$V_2 = I_2 Z_B + I_3 Z_B \tag{7.42}$$

$$0 = I_3(Z_A + Z_B + Z_C) - I_1 Z_A + I_2 Z_B \tag{7.43}$$

But the T circuit has only two loop equations given by:

$$V_1 = I_1(Z_1 + Z_3) + I_2 Z_3 \tag{7.44}$$

$$V_2 = I_1 Z_3 + I_2(Z_2 + Z_3) \tag{7.45}$$

We must eliminate one of the loop equations for the π circuit, and so we solve for I_3 in (7.43) and substitute the result into (7.41) and (7.42) to obtain:

$$V_1 = I_1\left[\frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C}\right] + I_2\left[\frac{Z_A Z_B}{Z_A + Z_B + Z_C}\right] \tag{7.46}$$

$$V_2 = I_1\left[\frac{Z_A Z_B}{Z_A + Z_B + Z_C}\right] + I_2\left[\frac{Z_B(Z_A + Z_C)}{Z_A + Z_B + Z_C}\right] \tag{7.47}$$

From a comparison of the coefficients of the currents in (7.46) and (7.47) with those in (7.44) and (7.45), we obtain the following relationships.

Replacing π with T

$$Z_1 = \frac{Z_A Z_C}{S_Z}; \quad Z_2 = \frac{Z_B Z_C}{S_Z}; \quad Z_3 = \frac{Z_A Z_B}{S_Z} \tag{7.48}$$

where

$$S_Z = Z_A + Z_B + Z_C$$

We can also replace a T network by a π network. To do this we need equations for Z_A, Z_B, and Z_C in terms of Z_1, Z_2, and Z_3. The required equations can be obtained algebraically from (7.48).

From T to π

$$Z_A = Z_1 + Z_3 + \frac{Z_1 Z_3}{Z_2}; \quad Z_B = Z_2 + Z_3 + \frac{Z_2 Z_3}{Z_1}; \quad Z_C = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3} \tag{7.49}$$

Reciprocity

If an LLFT network contains only Rs, Ls, Cs, and transformers but contains no dependent sources, then its loop impedance matrix is symmetrical with respect to the main diagonal. That is, if z_{ij} is an element of the loop impedance matrix (see (7.17)), occupying the position at row i and column j, then $z_{ji} = z_{ij}$, where z_{ji} occupies the position at row j and column i. Such a network has the property of reciprocity and is termed a reciprocal network.

Assume that a reciprocal network, depicted in Figure 7.46, has m loops and is in the zero state. It has only one excitation — a voltage source in loop j. To solve for the loop current in loop k, we write the loop equations:

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\ z_{21} & z_{22} & \cdots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j1} & z_{j2} & \cdots & z_{jm} \\ z_{k1} & z_{k2} & \cdots & z_{km} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mm} \end{bmatrix} \begin{bmatrix} I_2 \\ I_2 \\ \vdots \\ I_j \\ I_k \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ V_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.50}$$

The column excitation matrix has only one nonzero entry. To determine I_k using Cramer's rule, we replace column k by the excitation column and then expand along this column. Only one nonzero term is in the column, therefore, we obtain a single term for I_k:

$$I_k = V_j \frac{\Delta_{jk}}{\Delta} \tag{7.51}$$

where Δ_{jk} is the cofactor, and Δ is the determinant of the loop impedance matrix.

Next, we replace the voltage source by a short circuit in loop j, cut the wire in loop k, and insert a voltage source V_k. Figure 7.47 outlines the modifications to the circuit. Then, we solve for I_j obtaining

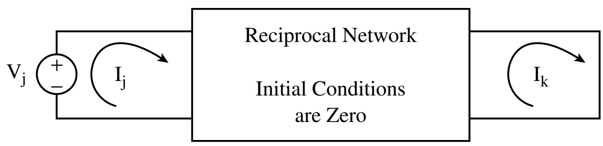


FIGURE 7.46 A reciprocal network with m independent loops.

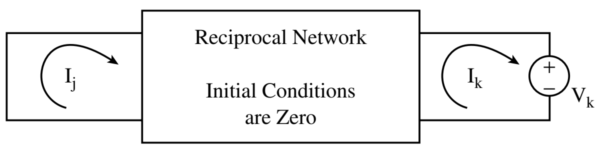


FIGURE 7.47 Interchange of the ports of excitation in the network in Figure 7.46.

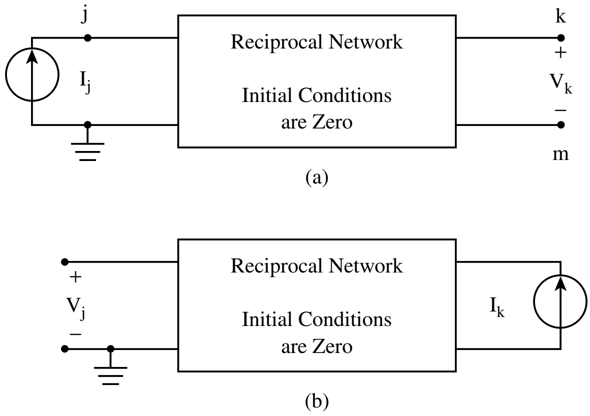


FIGURE 7.48 (a) Reciprocal ungrounded network with a current source excitation. (b) Interchange of the ports of excitation and response.

$$I_j = V_k \frac{\Delta_{kj}}{\Delta} \tag{7.52}$$

Because the network is reciprocal, $\Delta_{jk} = \Delta_{kj}$ so that

$$\frac{I_k}{V_j} = \frac{I_j}{V_k} \tag{7.53}$$

Eq. (7.53) is the statement of reciprocity for the network in Figures 7.46 and 7.47 with the excitations shown.

Figure 7.48(a) is a reciprocal network with a current excitation applied to node *j* and a voltage response, labeled V_k, taken between nodes *k* and *m*. We assume the network has *n* independent nodes plus the ground node indicated and is not a grounded network (does not have a common connection between the input and output ports shown). If we write node equations to solve for V_k in Figure 7.48(a) and use Cramer's rule, we have:

$$V_k = I_j \frac{\Delta'_{jk} - \Delta'_{jm}}{\Delta'} \tag{7.54}$$

where the primes indicate node-basis determinants. Then, we interchange the ports of excitation and response as depicted in Figure 7.48(b). If we solve for V_j in Figure 7.48(b), we obtain

$$V_j = I_k \frac{\Delta'_{kj} - \Delta'_{mj}}{\Delta'} \tag{7.55}$$

Because the corresponding determinants in (7.54) and (7.55) are equal because of reciprocity, we have:

$$\frac{V_k}{I_j} = \frac{V_j}{I_k} \tag{7.56}$$

Note that the excitations and responses are of the opposite type in Figures 7.46 and 7.48. The results obtained in (7.53) and (7.56) do not apply if the excitation and response are both voltages or both currents because when the ports of excitation and response are interchanged, the impedance levels of the network are changed [2].

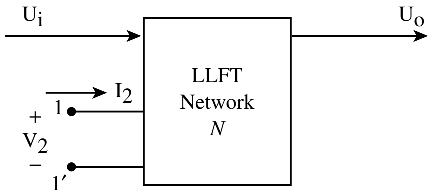


FIGURE 7.49 An arbitrary LLFT network.

Middlebrook's Extra Element Theorem

Middlebrook's extra element theorem is useful in developing tests for analog circuits and for predicting the effects that parasitic elements may have on a circuit. This theorem has two versions: the parallel version and the series version. Both versions present the results of connecting an extra network element in the circuit as the product of the network function obtained without the extra element times a correction factor. This is a particularly convenient form for the results because it shows exactly the effects of the extra element on the network function.

Parallel Version. Consider an arbitrary LLFT network having a transfer function $A_1(s)$. In the parallel version of the theorem, an impedance is added between any two independent nodes of the network. The modified transfer function is then obtained as $A_1(s)$ multiplied by a correction factor. Figure 7.49 is an arbitrary network in which the quantities U_i and U_0 represent a general input and a general output, respectively, whether they are voltages or currents. The extra element is to be connected between terminals 1 and 1′ in Figure 7.49, and the port variables for this port are V_2 and I_2.

We can write:

$$\begin{aligned} U_o &= A_1 U_i + A_2 I_2 \\ V_2 &= B_1 U_i + B_2 I_2 \end{aligned} \tag{7.57}$$

where

$$\begin{aligned} A_1 &= \left.\frac{U_o}{U_i}\right|_{I_2=0} \quad A_2 = \left.\frac{U_o}{I_2}\right|_{U_i=0} \\ B_1 &= \left.\frac{V_2}{U_i}\right|_{I_2=0} \quad B_2 = \left.\frac{V_2}{I_2}\right|_{U_i=0} \end{aligned} \tag{7.58}$$

Note that A_1 is assumed to be known.

The extra element Z to be added across terminals 1 and 1′ is depicted in Figure 7.50. It can be described as $Z = V_2/(-I_2)$ which yields $I_2 = V_2/(-Z)$. Substituting this expression for I_2 into (7.57) results in:

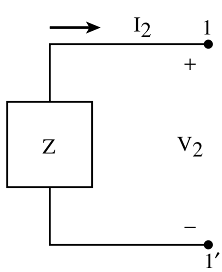


FIGURE 7.50 The extra element Z.

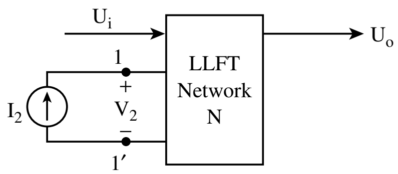


FIGURE 7.51 Network of Figure 7.49 with two excitations applied.

$$U_o = A_1 U_i + A_2\left(\frac{-V_2}{Z}\right)$$
$$V_2\left(1+\frac{B_2}{Z}\right) = B_1 U_i \quad (7.59)$$

After eliminating V_2 and solving for U_o/U_i, we obtain:

$$\frac{U_o}{U_i} = A_1\left[\frac{1+\frac{1}{Z}\left(\frac{A_1B_2 - A_2B_1}{A_1}\right)}{1+\frac{B_2}{Z}}\right] \quad (7.60)$$

Next, we provide physical interpretations for the terms in (7.60). Clearly, B_2 is the impedance seen looking into the network between terminals 1 and 1′ with $U_i = 0$. Thus, rename $B_2 = Z_d$ where d stands for "dead network."

To find a physical interpretation of $(A_1B_2 - A_2B_1)/A_1$, examine the network in Figure 7.51. Here, two excitations are applied to the network, namely U_i and I_2. Simultaneously adjust both inputs so as to null output U_o. Thus, with $U_o = 0$, we have from (7.57),

$$U_i = \frac{-A_2}{A_1} I_2 \quad (7.61)$$

Substituting this result into the equation for V_2 in (7.57), we have:

$$V_2 = B_1\left(\frac{-A_2}{A_1}\right)I_2 + B_2 I_2 \quad (7.62)$$

or

$$\left.\frac{V_2}{I_2}\right|_{U_o=0} = \frac{A_1B_2 - A_2B_1}{A_1}$$

Because the quantity $(A_1B_2 - A_2B_1)/A_1$ is the ratio of V_2 to I_2 with the output *nulled*, we rename this quantity as Z_N. Then rewriting (7.60) with Z_d and Z_N, we have:

$$\frac{U_o}{U_i} = A_1\left[\frac{1+\frac{Z_N}{Z}}{1+\frac{Z_d}{Z}}\right] \quad (7.63)$$

Equation (7.63) demonstrates that the results of connecting the extra element Z into the circuit can be expressed as the product of A_1, which is the network function with Z set to infinity, times a correction factor given in the brackets in (7.63).

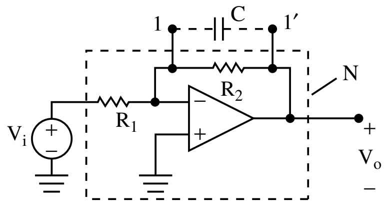


FIGURE 7.52 Ideal op amp circuit for Example 15.

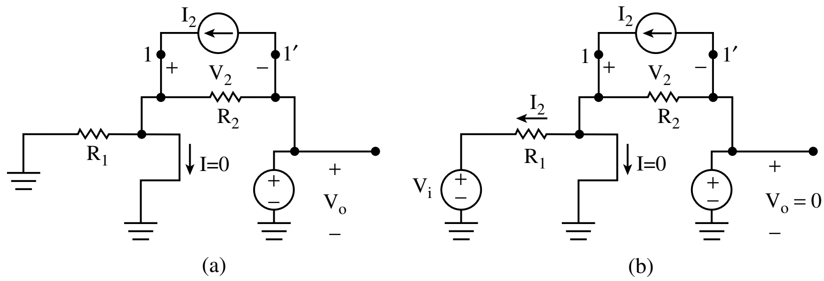


FIGURE 7.53 (a) Network for finding Z_d. (b) Network used to determine Z_N.

Example 15. Use the parallel version of Middlebrook's extra element theorem to find the voltage transfer function of the ideal op amp circuit in Figure 7.52 when a capacitor C is connected between terminals 1 and 1′.

Solution. With the capacitor not connected, the voltage transfer function is

$$\left.\frac{V_0}{V_i}\right|_{Z=\infty} = -\frac{R_2}{R_1} = A_1$$

Next, we determine Z_d from the circuit illustrated in Figure 7.53(a), where a model has been included for the ideal op amp, the excitation V_i has been properly removed, and a current excitation I_2 has been applied to the port formed by terminals 1 and 1′. Because no voltage flows across R_1 in Figure 7.53(a), no current flows through it, and all the current I_2 flows through R_2. Thus, $V_2 = I_2R_2$, and $Z_d = R_2$. We next find Z_N from Figure 7.53(b). We observe in this figure that the right end of R_2 is zero volts above ground because V_i and I_2 have been adjusted so that V_0 is zero. Furthermore, the left end of R_2 is zero volts above ground because of the virtual ground of the op amp. Thus, zero is current flowing through R_2, and so V_2 is zero. Consequently, $Z_N = V_2/I_2 = 0$. Following the format of (7.63), we have:

$$\frac{V_0}{V_i} = -\frac{R_2}{R_1}\left(\frac{1}{1+sCR_2}\right)$$

Note that for V_0 to be zero in Figure 7.53(b), V_i and I_2 must be adjusted so that $V_i/R_1 = -I_2$, although this information was not needed to work the example. □

Series Version. The series version of the theorem allows us to cut a loop of the network, add an impedance Z in series, and obtain the modified network function as $A_1(s)$ multiplied by a correction factor. A_1 is the network function when $Z = 0$. Figure 7.54 is an LLFT network with part of a loop illustrated explicitly. The quantities U_i and U_0 represent a general input and a general output, respectively, whether they be a voltage or a current. Define

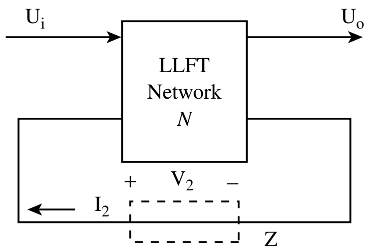


FIGURE 7.54 LLFT network used for the series version of Middlebrook's extra element theorem.

$$A_1 = \left.\frac{U_0}{U_i}\right|_{V_2=0} \quad A_2 = \left.\frac{U_0}{V_2}\right|_{U_i=0}$$
$$B_1 = \left.\frac{I_2}{U_i}\right|_{V_2=0} \quad B_2 = \left.\frac{I_2}{V_2}\right|_{U_i=0} \tag{7.64}$$

where V_2 and I_2 are depicted in Figure 7.54, and A_1 is assumed to be known. Then using superposition, we have:

$$U_0 = A_1U_i + A_2V_2$$
$$I_2 = B_1U_i + B_2V_2 \tag{7.65}$$

The impedance of the extra element Z can be described by $Z = V_2/(-I_2)$ so that $V_2 = -I_2Z$. Substituting this relation for V_2 into (7.65) and eliminating I_2, we have:

$$\frac{U_0}{U_i} = A_1\left[\frac{1 + Z\left(B_2 - B_1\dfrac{A_2}{A_1}\right)}{1 + B_2Z}\right] \tag{7.66}$$

Again, as we did for the parallel version of the theorem, we look for physical interpretations of the quantities in the square bracket in (7.66). From (7.65) we see that B_2 is the admittance looking into the port formed by cutting the loop in Figure 7.54 with $U_i = 0$. This is depicted in Figure 7.55(a). Thus, B_2 is the admittance looking into a dead network, and so let $B_2 = 1/Z_d$.

To find a physical interpretation of the quantity $(A_1B_2 - A_2B_1)/A_1$, we examine Figure 7.55(b) in which both inputs, V_2 and U_i, are adjusted to null the output U_0. From (7.65) with $U_o = 0$, we have:

$$U_i = -\frac{A_2}{A_1}V_2 \tag{7.67}$$

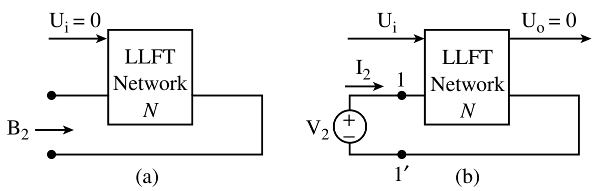


FIGURE 7.55 (a) Looking into the network with U_i equal zero. (b) U_i and V_2 are simultaneously adjusted to null the output U_0.

Then, eliminating U_i in (7.65) we obtain:

$$\frac{A_1 B_2 - A_2 B_1}{A_1} = \left.\frac{I_2}{V_2}\right|_{U_0=0} \tag{7.68}$$

Because this quantity is the admittance looking into the port formed by terminals 1 and 1′ in Figure 7.55(b) with U_o nulled, rename it as $1/Z_n$. Thus, from (7.66) we can write

$$\frac{U_o}{U_i} = A_1 \left[\frac{1 + \dfrac{Z}{Z_N}}{1 + \dfrac{Z}{Z_d}} \right] \tag{7.69}$$

Eq. (7.69) is particularly convenient for determining the effects of adding an impedance Z into a loop of a network.

Example 16. Use the series version of Middlebrook's extra element theorem to determine the effects of inserting a capacitor C in the location indicated in Figure 7.56.

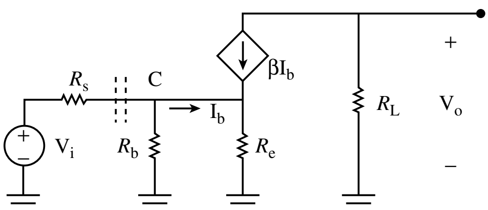


FIGURE 7.56 Network for Example 16.

Solution. The voltage transfer function for the network without the capacitor is found to be:

$$A_1 = \left.\frac{V_0}{V_i}\right|_{Z=0} = \frac{-\beta R_L}{R_s + (\beta+1) R_e \left(1 + \dfrac{R_s}{R_b}\right)}$$

Next, we find Z_d from Figure 7.57(a). This yields:

$$Z_d = \frac{V}{I} = R_s + \left[R_b \| (\beta+1) R_e \right]$$

The impedance Z_n is found from Figure 7.57(b) where the two input sources, V_i and V, are adjusted so that V_0 equals zero. If V_0 equals zero, then βI_b equals zero because no current flows through R_L. Thus, I_b equals zero, which implies that V_{Re}, the voltage across R_e as indicated in Figure 7.57(b), is also zero. We see that the null is propagating through the circuit. Continuing to analyze Figure 7.57(b), we see that I_{Rb} is zero so that we conclude that I is zero. Because $Z_N = V/I$, we conclude that Z_N is infinite. Using the format given by (7.69) with $Z = 1/(sC)$, we obtain the result as:

$$\frac{V_0}{V_i} = A_1 \left\{ \frac{1}{1 + \dfrac{1/(sC)}{R_s + \left[R_b \| (\beta+1) R_e \right]}} \right\}$$

□

It is interesting to note that to null the output so that Z_N could be found in Example 16, V_i is set to V, although this fact is not needed in the analysis.

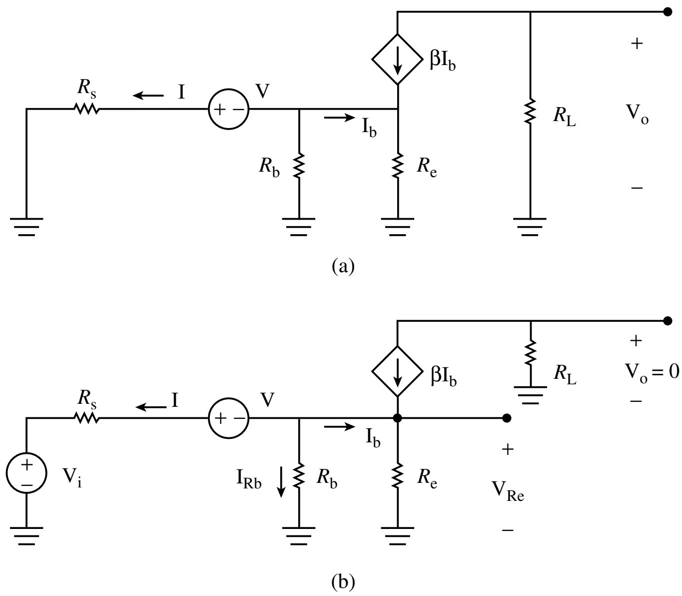


FIGURE 7.57 (a) Network used to obtain Z_d. (b) Network that yields Z_N.

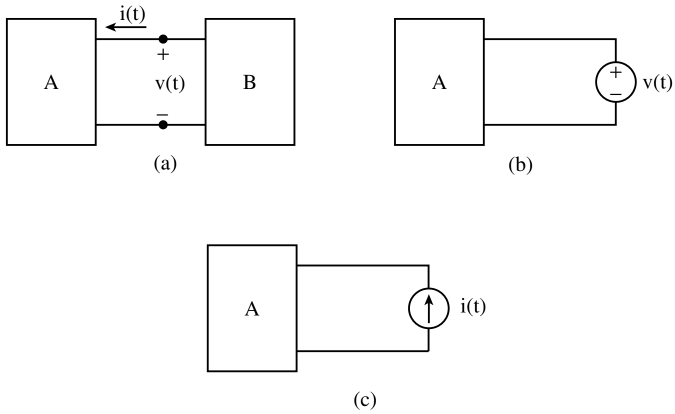


FIGURE 7.58 (a) An LLFT network consisting of two subnetworks A and B connected by two wires. (b) A voltage source can be substituted for subnetwork B if v(t) is known in (a). (c) A current source can be substituted for B if i is a known current.

Substitution Theorem

Figure 7.58(a) is an LLFT network consisting of two subnetworks A and B, which are connected to each other by two wires. If the voltage $v(t)$ is known, the voltages and currents in subnetwork A remain unchanged if a voltage source of value $v(t)$ is substituted for subnetwork B as illustrated in Figure 7.58(b).

Example 17. Determine $i_1(t)$ in the circuit in Figure 7.59. The voltage $v_1(t)$ is known from a previous analysis.

Solution. Because $v_1(t)$ is known, the substitution theorem can be applied to obtain the circuit in Figure 7.60. Analysis of this simplified circuit yields:

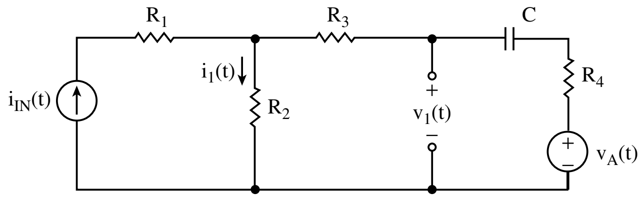


FIGURE 7.59 Circuit for Example 17.

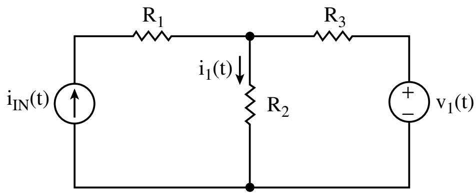


FIGURE 7.60 Circuit that results when the Substitution Theorem is applied to the circuit in Figure 7.59.

$$i_1 = i_{in}\frac{R_3}{R_2+R_3} + v_1\frac{1}{R_2+R_3}$$

If the current $i(t)$ is known in Figure 7.58(a), then the substitution in Figure 7.58(c) can be employed.

7.3 Sinusoidal Steady-State Analysis and Phasor Transforms

Sinusoidal Steady-State Analysis

In this section, we develop techniques for analyzing lumped, linear, finite, time invariant (LLFT) networks in the sinusoidal steady state. These techniques are important for analyzing and designing networks ranging from AC power generation systems to electronic filters.

To put the development of sinusoidal steady state analysis in its context, we list the following definitions of responses of circuits:

A. The **zero-input response** is the response of a circuit to its initial conditions when the input excitations are set to zero.
B. The **zero-state response** is the response of a circuit to a given input excitation or set of input excitations when the initial conditions are all set to zero.

The sum of the zero-input response and the zero-state response yields the total response of the system being analyzed. However, the total response can also be decomposed into the **forced response** and the **natural response** if the input excitations are DC, real exponentials, sinusoids, and/or sinusoids multiplied by real exponentials and if the exponent(s) in the input excitation differs from the exponents appearing in the zero-input response. These excitations are very common in engineering applications, and the decomposition of the response into forced and natural components corresponds to the particular and complementary (homogeneous) solutions, respectively, of the linear, constant coefficient, ordinary differential equations that characterize LLFT networks in the time domain. Therefore, we define:

C. The **forced response** is the portion of the total response that has the same exponents as the input excitations.
D. The **natural response** is the portion of the total response that has the same exponents as the zero-input response.

The sum of the forced and natural responses is the total response of the system.

For a strictly stable LLFT network, meaning that the poles of the system transfer function $T(s)$ are confined to the open left-half s-plane (LHP), the natural response must decay to zero eventually. The forced response may or may not decay to zero depending on the excitation and the network to which it is applied, and so it is convenient to define the terms **steady-state response** and **transient response**:

E. The **transient response** is the portion of the response that dies away or decays to zero with time.
F. The **steady-state response** is the portion of the response that does not decay with time.

The sum of the transient and steady state responses is the total response, but a specific circuit with a specific excitation may not have a transient response or it may not have a steady state response. The following example illustrates aspects of these six definitions.

Example 18. Find the total response of the network shown in Figure 7.61, and identify the zero-state, zero-input, forced, natural, transient, and steady-state portions of the response.

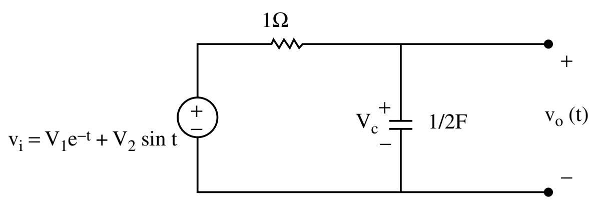


FIGURE 7.61 Circuit for Example 18.

Solution. Note that a nonzero initial condition is represented by V_c. Using superposition and the simple voltage divider concept, we can write:

$$V_0(s) = \left[\frac{V_1}{s+1} + \frac{V_2}{s^2+1}\right]\frac{2}{s+2} + \frac{V_c}{s}\left(\frac{s}{s+2}\right)$$

The partial fraction expansion for $V_0\ (s)$ is:

$$V_0(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}$$

where

$$A = 2V_1$$

$$B = -2V_1 + 0.4V_2 + V_c$$

$$C = -0.4V_2$$

$$D = 0.8V_2$$

Thus, for $t \geq 0$, $v_0\ (t)$ can be written as:

$$v_0(t) = 2V_1e^{-t} - 2V_1e^{-2t} + 0.4V_2e^{-2t} + V_ce^{-2t}$$
$$-0.4V_2\cos t + 0.8V_2\sin t$$

With the aid of the angle sum and difference formula

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

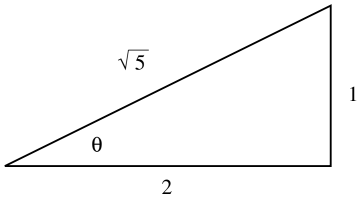


FIGURE 7.62 Sketch for determining the phase angle θ.

and the sketch in Figure 7.62, we can combine the last two terms in the expression for v_0 (t) to obtain:

$$v_0(t) = 2V_1e^{-t} - 2V_1e^{-2t} + 0.4V_2e^{-2t} + V_ce^{-2t}$$
$$+ 0.4\sqrt{5}V_2\sin(t+\theta)$$

where

$$\theta = -\tan^{-1}\left(\frac{1}{2}\right)$$

The terms of $v_0(t)$ are characterized by our definitions as follows:

$$\text{zero-input response} = V_ce^{-2t}$$
$$\text{zero-state response} = 2V_1e^{-t} - 2V_1e^{-2t} + 0.4V_2e^{-2t} + 0.4\sqrt{5}V_2\sin(t+\theta)$$
$$\text{natural response} = [-2V_1 + 0.4V_2 + V_c]e^{-2t}$$
$$\text{forced response} = 2V_1e^{-t} + 0.4\sqrt{5}V_2\sin(t+\theta)$$
$$\text{transient response} = 2V_1e^{-t} + [-2V_1 + 0.4V_2 + V_c]e^{-2t}$$
$$\text{steady-state response} = 0.4\sqrt{5}V_2\sin(t+\theta)$$

As can be observed by comparing the preceding terms above with the total response, part of the forced response is the steady state response, and the rest of the forced response is included in the transient response in this example. □

If the generator voltage in the previous example had been $v_i = V_1 + V_2$ sin(t), then there would have been two terms in the steady state response — a DC term and a sinusoidal term. On the other hand, if the transfer function from input to output had a pole at the origin and the excitation were purely sinusoidal, there would also have been a DC term and a sinusoidal term in the steady state response. The DC term would have arisen from the pole at the origin in the transfer function, and therefore would also be classed as a term in the natural response.

Oftentimes, it is desirable to obtain only the sinusoidal steady state response, without having to solve for other portions of the total response. The ability to solve for just the sinusoidal steady state response is the goal of sinusoidal steady state analysis.

The sinusoidal steady state response can be obtained based on analysis of the network using Laplace transforms. Figure 7.63 illustrates an LLFT network that is excited by the voltage sine wave $v_i(t) = V \sin(\omega t)$, where V is the peak amplitude of the sine wave and ω is the frequency of the sine wave in radians/second. Assume that the poles of the network transfer function $V_0(s)/V_i(s) = T(s)$ are confined to the open left-half s-plane (LHP) except possibly for a single pole at the origin. Then, the forced response of the network is

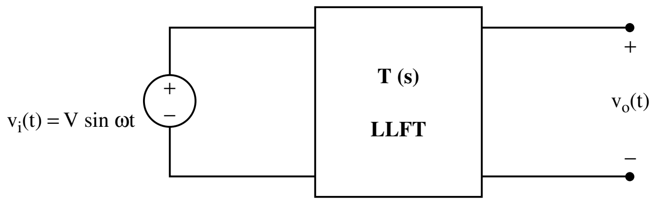


FIGURE 7.63 LLFT network with transfer function $T(s)$.

$$v_{oss}(t) = V|T(j\omega)|\sin(\omega t + \theta) \tag{7.70}$$

where the extra subscripts "*ss*" on $v_0(t)$ indicate sinusoidal steady state, and where

$$\theta = \tan^{-1}\left(\frac{\mathscr{I}T(j\omega)}{\mathscr{R}T(j\omega)}\right) \tag{7.71}$$

The symbols $\mathscr{I}$ and $\mathscr{R}$ are read as "imaginary part of" and "real part of," respectively. In other words, the LLFT network modifies the sinusoidal input signal in only two ways at steady state. The network multiplies the amplitude of the signal by $|T(j\omega)|$ and shifts the phase by θ. If the transfer function of the network is known beforehand, then the sinusoidal steady state portion of the total response can be easily obtained by means of Eqs. (7.70) and (7.71).

To prove (7.70) and (7.71), we assume that $T(s) = V_0(s)/V_i(s)$ in Figure 7.63 is real for s real and that the poles of $T(s)$ are confined to the open LHP except possibly for a single pole at the origin. Without loss of generality, assume the order of the numerator of $T(s)$ is at most one greater than the order of the denominator. Then the transform of the output voltage is

$$V_0(s) = V_i(s)T(s) = \frac{V\omega}{s^2 + \omega^2}T(s)$$

If $V_0(s)$ is expanded into partial fractions, we have:

$$V_0(s) = \frac{A}{s - j\omega} + \frac{B}{s + j\omega} + \text{other terms due to the poles of } T(s)$$

The residue A is

$$A = \left[(s - j\omega)V_o(s)\right]_{s=j\omega} = \left[\frac{V\omega}{s + j\omega}T(s)\right]_{s=j\omega}$$

$$= \frac{V}{2j}T(j\omega)$$

But

$$T(j\omega) = |T(j\omega)|e^{j\theta} \text{ where } \theta = \tan^{-1}\frac{\mathscr{I}T(j\omega)}{\mathscr{R}T(j\omega)}$$

Thus, we can write the residue A as

$$A = \frac{V}{2j}|T(j\omega)|e^{j\theta}$$

Also, $B = A^*$ where "*" denotes "conjugate," and so

$$B = -\frac{V}{2j}T(-j\omega) = -\frac{V}{2j}T\left((j\omega)^*\right)$$
$$= -\frac{V}{2j}T^*(j\omega) = -\frac{V}{2j}\left|T(j\omega)\right|e^{-j\theta}$$

In the equation for the residue B, we can write $T((j\omega)^*) = T^*(j\omega)$ because of the assumption that $T(s)$ is real for s real (see Property 1 in Section 7.1 on "Properties of LLFT Network Functions").

All other terms in the partial fraction of $V_0(s)$ will yield, when inverse transformed, functions of time that decay to zero except for a term arising from a pole at the origin of $T(s)$. A pole at the origin yields, when its partial fraction is inverse transformed, a DC term that is part of the steady-state solution in the time domain. However, only the first two terms in $V_0(s)$ will ultimately yield a sinusoidal function. We can rewrite these two terms as:

$$V_{oss}(s) = \frac{V}{2j}\left|T(j\omega)\right|\left[\frac{e^{j\theta}}{s - j\omega} - \frac{e^{-j\theta}}{s + j\omega}\right]$$

The extra subscripts "*ss*" denote sinusoidal steady state. The time-domain equation for the sinusoidal steady state output voltage is

$$v_{oss}(t) = \frac{V}{2j}\left|T(j\omega)\right|\left[e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t}\right]$$
$$= V\left|T(j\omega)\right|\sin(\omega t + \theta)$$

where θ is given by (7.71). This completes the proof.

Example 19. Verify the expression for the sinusoidal steady-state response found in the previous example.

Solution. The transfer function for the network in Figure 7.61 is $T(s) = 2/(s + 2)$, and the frequency of the sinusoidal portion of $v_i(t)$ is $\omega = 1$ rad/s. Thus,

$$T(j\omega) = \frac{2}{j+2} = \frac{2}{\sqrt{4+1}}e^{j\theta}$$

where

$$\theta = \tan^{-1}\left(\frac{-2}{4}\right) = -\tan^{-1}\left(\frac{1}{2}\right)$$

If the excitation in Figure 7.63 were $v_i(t) = V \sin(\omega t + \Phi)$, then the sinusoidal steady-state response of the network would be:

$$v_{oss}(t) = V\left|T(j\omega)\right|\sin(\omega t + \Phi + \theta) \tag{7.72}$$

where θ is given by (7.71). Similarly, if the excitation were $v_i(t) = V[\cos(\omega t + \Phi)]$, then the sinusoidal steady-state response would be expressed as:

$$v_{oss}(t) = V\left|T(j\omega)\right|\cos(\omega t + \Phi + \theta) \tag{7.73}$$

with θ again given by (7.71).

Phasor Transforms

In the sinusoidal steady-state analysis of stable LLFT networks, we find that both the inputs and outputs are sine waves of the same frequency. The network only modifies the amplitudes and the phases of the sinusoidal input signals; it does not change their nature. Thus, we need only keep track of the amplitudes and phases, and we do this by using phasor transforms. Phasor transforms are closely linked to Euler's identity:

$$e^{\pm j\omega t} = \cos(\omega t) \pm j\sin(\omega t) \tag{7.74}$$

If, for example, $v_i(t) = V\sin(\omega t + \Phi)$, then we can write $v_i(t)$ as

$$v_i(t) = \mathscr{I}\left[Ve^{j(\omega t+\Phi)}\right] = \mathscr{I}\left[Ve^{j\Phi}e^{j\omega t}\right] \tag{7.75}$$

Similarly, if $v_i(t) = V\cos(\omega t + \Phi)$, then we can write

$$v_i(t) = \mathscr{R}\left[Ve^{j(\omega t+\Phi)}\right] = \mathscr{R}\left[Ve^{j\Phi}e^{j\omega t}\right] \tag{7.76}$$

If we confine our analysis to single-frequency sine waves, then we can drop the imaginary sign and the term $e^{j\omega t}$ in (7.75) to obtain the phasor transform. That is,

$$\wp\left[v_i(t)\right] = \wp\left[V\sin(\omega t + \Phi)\right] = \wp\left\{\mathscr{I}\left[Ve^{j\Phi}e^{j\omega t}\right]\right\} = Ve^{j\Phi} \tag{7.77}$$

The first and last terms in (7.77) are read as "the phasor transform of $v_i(t)$ equals $Ve^{j\Phi}$". Note that $v_i(t)$ is not equal to $Ve^{j\Phi}$ as can be seen from the fact that $v_i(t)$ is a function of time while $Ve^{j\Phi}$ is not. Phasor transforms will be denoted with **bold** letters that are underlined as in $\wp[v_i(t)] = \underline{\boldsymbol{V}}_i$.

If our analysis is confined to single-frequency cosine waves, we perform the phasor transform in the following manner:

$$\wp\left[V\cos(\omega t + \Phi)\right] = \wp\left\{\mathscr{R}\left[Ve^{j\Phi}e^{j\omega t}\right]\right\} = Ve^{j\Phi} = \underline{\boldsymbol{V}} \tag{7.78}$$

In other words, to perform the phasor transform of a cosine function, we drop both the real sign and the term $e^{j\omega t}$. Both sines and cosines are sinusoidal functions, but when we transform them, they lose their identities. Thus, before starting an analysis, we must decide whether to perform the analysis all in sines or all in cosines. The two functions must not be mixed when using phasor transforms. Furthermore, we cannot simultaneously employ the phasor transforms of sinusoids at two different frequencies. However, if a linear network has two excitations which have different frequencies, we can use superposition in an analysis for a voltage or current, and add the solutions in the time domain.

Three equivalent representations are used for a phasor $\underline{\boldsymbol{V}}$:

$$\underline{\boldsymbol{V}} = \begin{cases} Ve^{j\Phi} & \text{exponential form} \\ V(\cos\Phi + j\sin\Phi) & \text{rectangular form} \\ V\angle\Phi & \text{polar form} \end{cases} \tag{7.79}$$

If phasors are to be multiplied or divided by a complex number, the exponential or polar forms are the most convenient. If phasors are to be added or subtracted, the rectangular form is the most convenient. The relationships among the equivalent representations are illustrated in Figure 7.64. In this figure, the phasor $\underline{\boldsymbol{V}}$ is denoted by a point in the complex plane. The magnitude of the phasor, $|\underline{\boldsymbol{V}}| = V$, is illustrated

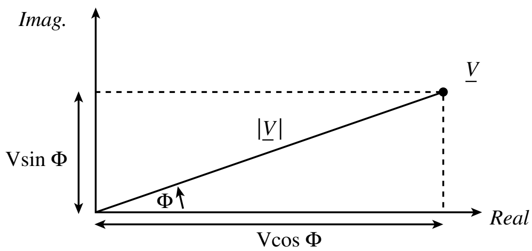


FIGURE 7.64 Relationships among phasor representations.

by the length of the line drawn from the origin to the point. The phase of the phasor, Φ, is shown measured counterclockwise from the horizontal axis. The real part of $\underline{V}$ is $V\cos\Phi$ and the imaginary part of $\underline{V}$ is $V\sin\Phi$.

Phasors can be developed in a way that parallels, to some extent, the usual development of Laplace transforms. In the following theorems, we assume that the constants V_1, V_2, Φ_1, and Φ_2 are real.

Theorem 1: *For sinusoids of the same type (either sines or cosines) and of the same frequency* ω, $\wp[V_1\sin(\omega t+\Phi_1)+V_2\sin(\omega t+\Phi_2)] = V_1\wp[\sin(\omega t+\Phi_1)] + V_2\wp[\sin(\omega t+\Phi_2)]$. *A similar relation can be written for cosines.*

This theorem demonstrates that the phasor transform is a linear transform.

Theorem 2: *If* $\wp[V_1\sin(\omega t+\Phi)] = V_1e^{j\Phi}$, *then*

$$\wp\left[\frac{d}{dt}V_1\sin(\omega t+\Phi)\right] = j\omega V_1e^{j\Phi} \tag{7.80}$$

To prove Theorem 2, we can write:

$$\begin{aligned}\wp\left[\frac{d}{dt}V_1\mathcal{I}\left(e^{j\Phi}e^{j\omega t}\right)\right] &= \wp\left[V_1\mathcal{I}\left(e^{j\Phi}\frac{d}{dt}e^{j\omega t}\right)\right]\\ &= \wp\left[V_1\mathcal{I}e^{j\Phi}j\omega e^{j\omega t}\right] = V_1j\omega e^{j\Phi}\end{aligned}$$

Note the interchange of the derivative and the imaginary sign in the proof of the theorem. Also, Theorem 2 can be generalized to:

$$\wp\left[\frac{d^n}{dt^n}V_1\sin(\omega t+\Phi)\right] = (j\omega)^nV_1e^{j\Phi} \tag{7.81}$$

These results are useful for finding the sinusoidal steady state solutions of linear, constant-coefficient, ordinary differential equations assuming the roots of the characteristic polynomials lie in the open LHP with possibly one at the origin.

Theorem 3: *If* $\wp[V_1\sin(\omega t+\Phi)] = V_1e^{j\Phi}$, *then*

$$\wp\left[\int V_1\sin(\omega t+\Phi)\,dt\right] = \frac{1}{j\omega}V_1e^{j\Phi} \tag{7.82}$$

The proof of Theorem 3 is easily obtained by writing:

$$\wp\left[\int V_1 \sin(\omega t + \Phi)\, dt\right] = \wp\left[\int \mathscr{I}\left[V_1 e^{j(\omega t+\Phi)}\right] dt\right]$$
$$= \wp\left[\mathscr{I}\int V_1 e^{j(\omega t+\Phi)}\, dt\right] = \frac{V_1}{j\omega} e^{j\Phi}$$

It should be noted that no constant of integration is employed because a constant is not a sinusoidal function and is therefore not permitted when using phasors. A constant of integration arises in LLFT network analysis because of initial conditions, and we are interested only in the sinusoidal steady-state response and not in a zero-input response. No limits are used with the integral either, because the (constant) lower limit would also yield a constant, which would imply that we are not at sinusoidal steady state.

Theorem 3 is easily extended to the case of n integrals:

$$\wp\left[\int \ldots \int V_1 \sin(\omega t + \Phi)(dt)^n\right] = \frac{V_1}{(j\omega)^n} e^{j\Phi} \tag{7.83}$$

This result is useful for finding the sinusoidal steady-state solution of integro-differential equations.

Inverse Phasor Transforms

To obtain time domain results, we must be able to inverse transform phasors. The inverse transform operation is denoted by $\wp^{-1}$. This is an easy operation that consists of restoring the term $e^{j\omega t}$, restoring the imaginary sign (the real sign if cosines are used), and dropping the inverse transform sign. That is,

$$\begin{aligned}\wp^{-1}\left[V_1 e^{j\Phi}\right] &= \mathscr{I}\left[V_1 e^{j\Phi} e^{j\omega t}\right] \\ &= V_1 \sin(\omega t + \Phi)\end{aligned} \tag{7.84}$$

The following example illustrates both the use of Theorem 2 and the inverse transform procedure.

Example 23. Determine the sinusoidal steady-state solution for the differential equation:

$$\frac{d^2 f(t)}{dt^2} + 4\frac{df(t)}{dt} + 3f(t) = V \sin(\omega t + \Phi)$$

Solution. We note that the characteristic polynomial, $D^2 + 4D + 3$, has all its roots in the open LHP. The next step is to phasor transform each term of the equation to obtain:

$$-\omega^2 \underline{\boldsymbol{F}} + 4j\omega\underline{\boldsymbol{F}} + 3\underline{\boldsymbol{F}} = Ve^{j\Phi}$$

where $\underline{\boldsymbol{F}}(j\omega) = \wp[f(t)]$. Therefore, when we solve for $\underline{\boldsymbol{F}}$, we obtain

$$\begin{aligned}\underline{\boldsymbol{F}} &= \frac{Ve^{j\Phi}}{(3-\omega^2) + j4\omega} \\ &= \frac{Ve^{j\Phi}e^{j\theta}}{\sqrt{(3-\omega^2)^2 + 16\omega^2}}\end{aligned}$$

where

$$\theta = \tan^{-1}\frac{-4\omega}{3-\omega^2} = \tan^{-1}\frac{4\omega}{\omega^2 - 3}$$

Thus,

$$\underline{\boldsymbol{F}} = \frac{V}{\sqrt{\omega^4 + 10\omega^2 + 9}} e^{j(\phi+\theta)}$$

To obtain a time-domain function, we inverse transform $\underline{\boldsymbol{F}}$ to obtain:

$$\wp^{-1}\left[\underline{\boldsymbol{F}}(j\omega)\right] = f(t) = \frac{V}{\sqrt{\omega^4 + 10\omega^2 + 9}} \sin(\omega t + \Phi + \theta)$$

In this example, we see that the sinusoidal steady-state solution consists of the sinusoidal forcing term, $V \sin(\omega t + \Phi)$, modified in amplitude and shifted in phase.

Phasors and Networks

Phasors are time-independent representations of sinusoids. Thus, we can define impedances in the phasor transform domain and obtain Ohm's law-like expressions relating currents through network elements with the voltages across those elements. In addition, the impedance concept allows us to combine dissimilar elements, such as resistors with inductors, in the transform domain.

The time-domain expressions relating the voltages and currents for *Rs*, *Ls*, and *Cs*, repeated here for convenience, are:

$$v_R(t) = i_R(t)R \qquad v_L(t) = \frac{Ldi_L}{dt} \qquad v_c(t) = \frac{1}{C}\int i_C dt$$

Note that initial conditions are set to zero. Then, performing the phasor transform of the time-domain variables, we have

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$$

We can also write the admittances of these elements as $Y_R = 1/Z_R$, $Y_L = 1/Z_L$, and $Y_C = 1/Z_C$. Then, we can extend the impedance and admittance concepts for two-terminal elements to multiport networks in the same manner as was done in the development of Laplace transform techniques for network analysis. For example, the transfer function of the circuit shown in Figure 7.65 can be written as:

$$\frac{\underline{\boldsymbol{V}}_0(j\omega)}{\underline{\boldsymbol{V}}_i(j\omega)} = G_{21}(j\omega)$$

where the "$j\omega$" indicates that the analysis is being performed at sinusoidal steady state [1]. It is also assumed that no other excitations exist in *N* in Figure 7.65. With impedances and transfer functions defined, then all the theorems developed for Laplace transform analysis, including source transformations, have a phasor transform counterpart.

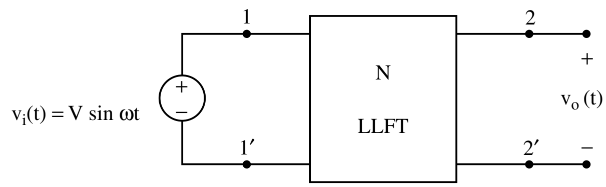


FIGURE 7.65 An LLFT network excited by a sinusoidal voltage source.

Example 21. Use phasor analysis to find the transfer function G_{21} (jω) and v_{oss} (t) for the circuit in Figure 7.66.

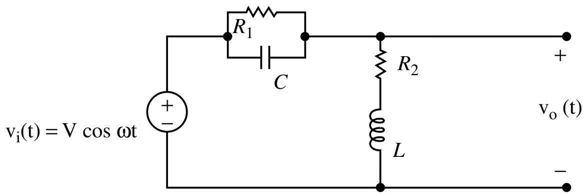


FIGURE 7.66 Circuit for Example 21.

Solution. The phasor transform of the output voltage can be obtained easily by means of the simple voltage divider. Thus,

$$\underline{V}_o = \underline{V}_i \frac{j\omega L + R_2}{j\omega L + R_2 + \dfrac{R_1}{1 + j\omega C R_1}}$$

To obtain G_{21}(jω), we form $\underline{V}_0/\underline{V}_i$, which yields

$$\frac{\underline{V}_0}{\underline{V}_i} = G_{21}(j\omega) = \frac{(R_2 + j\omega L)(1 + j\omega C R_1)}{(R_2 + j\omega L)(1 + j\omega C R_1) + R_1}$$

Expressing the numerator and denominator of G_{21} in exponential form produces:

$$G_{21} = \frac{\sqrt{(R_2 - \omega^2 LCR_1)^2 + (\omega L + \omega C R_1 R_2)^2}\, e^{j\alpha}}{\sqrt{(R_1 + R_2 - \omega^2 LCR_1)^2 + (\omega L + \omega C R_1 R_2)^2}\, e^{j\beta}}$$

where

$$\alpha = \tan^{-1} \frac{(\omega L + \omega C R_1 R_2)}{R_2 - \omega^2 LCR_1}$$

$$\beta = \tan^{-1} \frac{(\omega L + \omega C R_1 R_2)}{R_1 + R_2 - \omega^2 LCR_1}$$

Thus,

$$G_{21}(j\omega) = Me^{j\theta}$$

where

$$M = \sqrt{\frac{(R_2 - \omega^2 LCR_1)^2 + (\omega L + \omega C R_1 R_2)^2}{(R_1 + R_2 - \omega^2 LCR_1)^2 + (\omega L + \omega C R_1 R_2)^2}}$$

and

$$\theta = \alpha - \beta$$

The phasor transform of $v_i(t)$ is

$$\underline{\boldsymbol{V}}_i = \wp\left[V\mathcal{R}e^{j\omega t}\right] = Ve^{j0} = V$$

and, therefore, the time-domain expression for the sinusoidal steady-state output voltage is:

$$v_{\text{oss}}(t) = VM\cos(\omega t + \theta) \qquad \square$$

Driving point impedances and admittances as well as transfer functions are not phasors because they do not represent sinusoidal waveforms. However, an impedance or transfer function is a complex number at a particular real frequency, and the product of a complex number times a phasor is a new phasor.

The product of two arbitrary phasors is not ordinarily defined because $\sin^2(\omega t)$ or $\cos^2(\omega t)$ are not sinusoidal and have no phasor transforms. However, as we will see later, power relations for AC circuits can be expressed in efficient ways as functions of products of phasors. Because such products have physical interpretations, we permit them in the context of power calculations.

Division of one phasor by another is permitted only if the two phasors are related by a driving point or transfer network function such as $\underline{\boldsymbol{V}}_0/\underline{\boldsymbol{V}}_i = G_{21}(j\omega)$.

Phase Lead and Phase Lag

The terms "phase lead" and "phase lag" are used to describe the phase shift between two or more sinusoids of the same frequency. This phase shift can be expressed as an angle in degrees or radians, or it can be expressed in time as seconds. For example, suppose we have three sinusoids given by:

$$v_1(t) = V_1\sin(\omega t) \qquad v_2(t) = V_2\sin(\omega t + \Phi) \qquad v_3 = V_3\sin(\omega t - \Phi)$$

where V_1, V_2, V_3, and Φ are all positive. Then, we say that v_2 **leads** v_1 and that v_3 **lags** v_1. To see this more clearly, we rewrite v_2 and v_3 as:

$$v_2 = V_2\sin[\omega(t + t_0)] \qquad\qquad v_3 = V_3\sin[\omega(t - t_0)]$$

where the constant $t_0 = \Phi/\omega$. Figure 7.67 plots the three sinusoids sketched on the same axis, and from this graph we see that the zero crossings of $v_2(t)$ occur t_0 seconds before the zero crossing of $v_1(t)$. Thus, $v_2(t)$ leads $v_1(t)$ by t_0 seconds. Similarly, we see that the zero crossings of $v_3(t)$ occur t_0 seconds after the zero crossings of $v_1(t)$. Thus, $v_3(t)$ lags $v_1(t)$. We can also say that $v_3(t)$ lags $v_2(t)$. When comparing the phases of sine waves with $V\sin(\omega t)$, the key thing to look for in the arguments of the sines are the signs of the angles following ωt. A positive sign means lead and a negative sign means lag. If two sines or two cosines have the same phase angle, then they are called "in phase."

If we have $i_1(t) = I_1[\cos(\omega t - \pi/4)]$ and $i_2(t) = I_2[\cos(\omega t - \pi/3)]$, then i_2 lags i_1 by $\pi/12$ rad or 15° because even though the phases of both cosines are negative, the phase of $i_1(t)$ is less negative than the phase of $i_2(t)$. We can also say that i_1 leads i_2 by 15°.

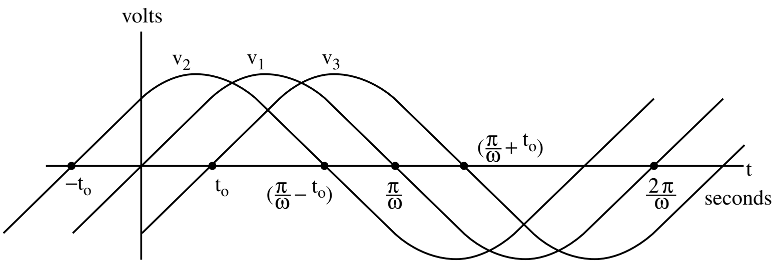


FIGURE 7.67 Three sinusoids sketched on a time axis.

Example 22. Suppose we have five signals with equal peak amplitudes and equal frequencies but with differing phases. The signals are: $i_1 = I\,[\sin(\omega t)]$, $i_2 = I\,[\cos(\omega t)]$, $i_3 = I\,[\cos(\omega t + \theta)]$, $i_4 = -I\,[\sin(\omega t + \psi)]$, and $i_5 = -I\,[\cos(\omega t - \Phi)]$. Assume I, θ, ψ, and Φ are positive.

A. How much do the signals i_2 through i_5 lead i_1?
B. How much do the signals i_1 and i_3 through i_5 lead i_2?

Solution. For part (A), we express i_2 through i_5 as sines with lead angles. That is,

$$i_2 = I\cos(\omega t) = I\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$i_3 = I\cos(\omega t + \theta) = I\sin\left(\omega t + \theta + \frac{\pi}{2}\right)$$

$$i_4 = -I\sin(\omega t + \psi) = I\sin(\omega t + \psi \pm \pi)$$

$$i_5 = -I\cos(\omega t - \Phi) = I\cos(\omega t - \Phi \pm \pi)$$

$$= I\sin\left(\omega t - \Phi \pm \pi + \frac{\pi}{2}\right)$$

Thus, i_2 leads i_1 by $\pi/2$ rad, and i_3 leads i_1 by $\theta + \pi/2$. For i_4, we can take the plus sign in the argument of the sign to obtain $\psi + \pi$, or we can take the minus sign to obtain $\psi - \pi$. The current i_5 leads i_1 by $(3\pi/2 - \Phi)$ or by $(-\pi/2 - \Phi)$. An angle of $\pm 2\pi$ can be added to the argument without affecting lead or lag relationships.

For part (B), we express i_1 and i_3 through i_5 as cosines with lead angles yielding:

$$i_1 = I\sin(\omega t) = I\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$i_3 = I\cos(\omega t + \theta)$$

$$i_4 = -I\sin(\omega t + \psi) = I\sin(\omega t + \psi \pm \pi)$$

$$= I\cos\left(\omega t + \psi \pm \pi - \frac{\pi}{2}\right)$$

$$i_5 = -I\cos(\omega t - \Phi) = I\cos(\omega t - \Phi \pm \pi)$$

We conclude that i_1 leads i_2 by $(-\pi/2)$ rad. (We could also say that i_1 lags i_2 by $(\pi/2)$ rad.) Also, i_3 leads i_2 by θ. The current i_4 leads i_2 by $(\psi + \pi/2)$ where we have chosen the plus sign in the argument of the cosine. Finally, i_5 leads i_2 by $(\pi - \Phi)$, where we have chosen the plus sign in the argument. □

In the previous example, we have made use of the identities:

$$\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right); \qquad -\sin(\alpha) = \sin(\alpha \pm \pi)$$

$$-\cos(\alpha) = \cos(\alpha \pm \pi); \qquad \sin(\alpha) = \cos\left(\alpha - \frac{\pi}{2}\right)$$

The concepts of phase lead and phase lag are clearly illustrated by means of phasor diagrams, which are described in the next section.

Phasor Diagrams

Phasors are complex numbers that represent sinusoids, so phasors can be depicted graphically on a complex plane. Such graphical illustrations are called phasor diagrams. Phasor diagrams are valuable because they present a clear picture of the relationships among the currents and voltages in a network. Furthermore, addition and subtraction of phasors can be performed graphically on a phasor diagram. The construction of phasor diagrams is demonstrated in the next example.

Example 23. For the network in Figure 7.68(a), find $\underline{\boldsymbol{I}}_1$, $\underline{\boldsymbol{V}}_{R1}$, and $\underline{\boldsymbol{V}}_C$. For Figure 7.68(b), find $\underline{\boldsymbol{I}}_2$, $\underline{\boldsymbol{V}}_{R2}$, and $\underline{\boldsymbol{V}}_L$. Construct phasor diagrams that illustrate the relations of the currents to the voltage excitation and the other voltages of the networks.

Solution. For Figure 7.68(a), we have

$$\wp[v(t)] = V\angle 0^\circ \quad \text{and} \quad \wp[i_1(t)] = \underline{\boldsymbol{I}}_1 = \frac{V}{R_1 + \dfrac{1}{j\omega C}}$$

Rewriting $\underline{\boldsymbol{I}}_1$, we have:

$$\underline{\boldsymbol{I}}_1 = \frac{Vj\omega C}{1 + j\omega CR_1} = \frac{Vj\omega C}{1 + j\omega CR_1}\left[\frac{1 - j\omega CR_1}{1 - j\omega CR_1}\right]$$

$$= V\left[\frac{\omega^2C^2R_1 + j\omega C}{\omega^2C^2R_1^2 + 1}\right] = \frac{V\omega C}{\sqrt{\omega^2C^2R_1^2 + 1}}e^{j\theta_1}$$

where

$$\theta_1 = \tan^{-1}\frac{\omega C}{\omega^2C^2R_1} = \tan^{-1}\frac{1}{\omega CR_1}$$

Note that we have multiplied the numerator and denominator of $\underline{\boldsymbol{I}}_1$ by the conjugate of the denominator. The resulting denominator of $\underline{\boldsymbol{I}}_1$ is purely real, and so we need only consider the terms in the numerator of $\underline{\boldsymbol{I}}_1$ to obtain an expression for the phase. Thus, the resulting expression for the phase contains only one term which has the form:

$$\theta_1 = \tan^{-1}\frac{\mathcal{I}(\text{numerator})}{\mathcal{R}(\text{numerator})}$$

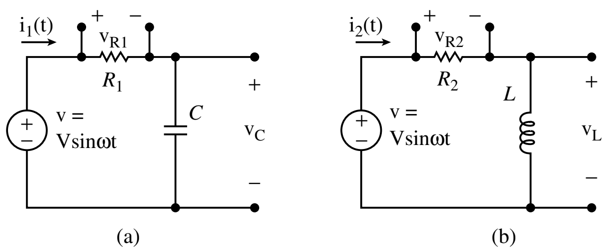


FIGURE 7.68 (a) An RC network. (b) An RL network.

We could have obtained the same results without application of this artifice. In this case, we would have obtained

$$\theta_1 = \frac{\pi}{2} - \tan^{-1}\omega CR_1$$

For $\omega CR_1 \geq 0$, it is easy to show that the two expressions for θ_1 are equivalent.

Because the same current flows through both network elements, we have

$$\underline{\boldsymbol{V}}_{R1} = \frac{V\omega CR_1}{\sqrt{\omega^2C^2R_1^2+1}}e^{j\theta_1}$$

and

$$\underline{\boldsymbol{V}}_{c} = \underline{\boldsymbol{I}}_1\left(\frac{1}{j\omega C}\right) = \frac{-jV}{\sqrt{\omega^2C^2R_1^2+1}}e^{j\theta_1} = \frac{V}{\sqrt{\omega^2C^2R_1^2+1}}e^{j\psi}$$

where

$$\psi = -\frac{\pi}{2} + \theta_1 = -\tan^{-1}\omega CR_1$$

For $\underline{\boldsymbol{I}}_2$ in Figure 7.68(b), we obtain

$$\underline{\boldsymbol{I}}_2 = \frac{V\angle 0^o}{R_2 + j\omega L} = \frac{V}{\sqrt{R_2^2 + w^2L^2}}e^{j\theta_2}$$

where θ_2 is given by

$$\theta_2 = -\tan^{-1}\frac{\omega L}{R_2}$$

The phasor current $\underline{\boldsymbol{I}}_2$ flows through both R_2 and L. So we have:

$$\underline{\boldsymbol{V}}_{R2} = \underline{\boldsymbol{I}}_2R_2$$

and

$$\underline{\boldsymbol{V}}_{L} = j\omega L\underline{\boldsymbol{I}}_2 = \frac{V\omega L}{\sqrt{\omega^2L^2+R_2^2}}e^{j\Phi}$$

where

$$\Phi = \frac{\pi}{2} + \theta_2$$

To construct the phasor diagram in Figure 7.69(a) for the RC network in Figure 7.68(a), we first draw a vector corresponding to the phasor transform $\underline{\boldsymbol{V}} = V\angle 0°$ of the excitation. Because the phase of this phasor is zero, it is represented as a vector along the positive real axis. The length of this vector is $|\underline{\boldsymbol{V}}|$. Then we construct the vector representing $\underline{\boldsymbol{I}}_1 = |\underline{\boldsymbol{I}}_1|e^{j\theta 1}$. Again, the length of the vector is $|\underline{\boldsymbol{I}}_1|$, and it is

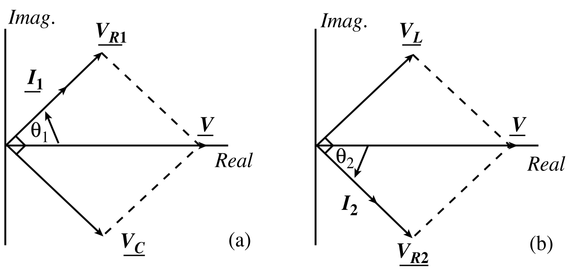


FIGURE 7.69 (a) Phasor diagram for the voltages and currents in Figure 7.68(a). (b) Phasor diagram for Figure 7.68(b).

drawn at the angle θ_1. The vector representing $\underline{V}_{R1}$ lies along $\underline{I}_1$ because the voltage across a resistor is always in phase or 180° out of phase with the current flowing through the resistor. The vector representing the current leads $\underline{V}_C$ by exactly 90°. It should be noted from the phasor diagram that $\underline{V}_{R1}$ and $\underline{V}_C$ add to produce $\underline{V}$ as required by Kirchhoff's law. □

Figure 7.69(b) presents the phasor diagram for the RL network in Figure 7.68(b). For this network, $\underline{I}_2$ lags $\underline{V}_L$ by exactly 90°. Also, the vector sum of the voltages $\underline{V}_L$ and $\underline{V}_{R2}$ must be the excitation voltage $\underline{V}$ as indicated by the dotted lines in Figure 7.69(b).

If the excitation $V\sin(\omega t)$ had been $V\sin(\omega t + \Phi)$ in Figure 7.68 in the previous example, then the vectors in the phasor diagrams in Figure 7.69 would have just been rotated around the origin by Φ. Thus, for example, $\underline{I}_1$ in Figure 7.69(a) would have an angle equal to $\theta_1 + \Phi$. The lengths of the vectors and the relative phase shifts between the vectors would remain the same.

If R_1 in Figure 7.68(a) is decreased, then from the expression for $\theta_1 = \tan^{-1}(1/(\omega C R_1))$, we see that the phase of $\underline{I}_1$ is increased. As R_1 is reduced further, θ_1 approaches 90°, and the circuit becomes more nearly like a pure capacitor. However, as long as $\underline{I}_1$ leads $\underline{V}$, we label the circuit as capacitive.

As R_2 in Figure 7.68(b) is decreased, then θ_2 in Figure 7.69(b) decreases (becomes more negative) and approaches –90°. Nevertheless, as long as $\underline{I}_2$ lags $\underline{V}$, we refer to the circuit as inductive.

If both inductors and capacitors are in a circuit, then it is possible for the circuit to appear capacitive at some frequencies and inductive at others. An example of such a circuit is provided in the next section.

Resonance

Resonant networks come in two basic types: the parallel resonant network and the series resonant (sometimes called antiresonant) network. More complicated networks may contain a variety of both types of resonant circuits. To see what happens at resonance, we examine a parallel resonant network at sinusoidal steady state [1]. Figure 7.70 is a network consisting of a capacitor and inductor connected in parallel, often called a tank circuit or tank, and an additional resistor R_1 connected in parallel with the tank. The phasor transforms of the excitation and the currents through the elements in Figure 7.70 are:

$$\overline{V} = V\angle 0^\circ;\ \underline{I}_{R1} = \frac{V}{R_1};\ \underline{I}_C = j\omega C V;\ \underline{I}_L = \frac{V}{j\omega L} \tag{7.85}$$

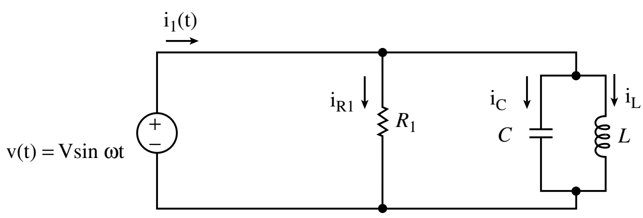


FIGURE 7.70 Parallel resonant circuit.

where V is the peak value of the excitation. The transform of the current supplied by the source is

$$\underline{I_1} = I_1 \angle\theta_1 = \underline{I_{R1}} + \underline{I_C} + \underline{I_L} = V\left[\frac{1}{R_1} + j\omega C\left(1 - \frac{1}{\omega^2 LC}\right)\right] \tag{7.86}$$

The peak value of the current $i_1(t)$ at steady state is

$$I_1 = V\sqrt{\left(\frac{1}{R_1}\right)^2 + \omega^2 C^2\left(1 - \frac{1}{\omega^2 LC}\right)^2} \tag{7.87}$$

It is not difficult to determine that the minimum value of I_1 occurs at

$$\omega = \frac{1}{\sqrt{LC}} \tag{7.88}$$

which is the condition for resonance, and $I_{1\min}$ is given by

$$I_{1\min} = \frac{V}{R_1} \tag{7.89}$$

This result is somewhat surprising since it means that at resonance, the source in Figure 7.70 delivers no current to the tank at steady state. However, this result does not mean that the currents through the capacitor and inductor are zero. In fact, for $\omega^2 = 1/(LC)$ we have:

$$\underline{I_C} = jV\sqrt{\frac{C}{L}} \text{ and } \underline{I_L} = -jV\sqrt{\frac{C}{L}}$$

That is, the current through the inductor is 180° out of phase with the current through the capacitor, and, because their magnitudes are equal, their sum is zero. Thus, at steady state and at the frequency given by (7.88), the tank circuit looks like an open circuit to the voltage source. Yet, a circulating current occurs in the tank, labeled $\underline{I_T}$ in Figure 7.71, which can be quite large depending on the values of C and L. That is, at resonance,

$$\underline{I_T} = jV\sqrt{\frac{C}{L}} = \underline{I_C} = -\underline{I_L} \tag{7.90}$$

Therefore, energy is being transferred back and forth between the inductor and the capacitor. If the inductor and capacitor are ideal, the energy transferred would never decrease. In practice, parasitic resistances, especially in a physical inductor, would eventually dissipate this energy. Of course, parasitic resistances can be modeled as additional elements in the network.

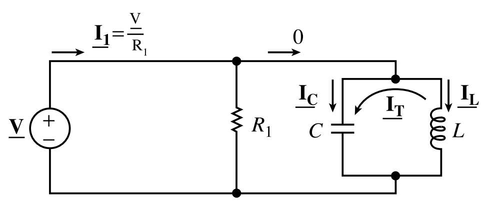


FIGURE 7.71 Circuit of Figure 7.70 at resonance. No current is supplied to the tank by the source, but a circulating current occurs in the tank.

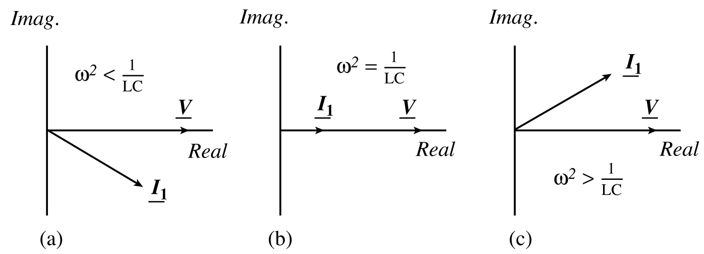


FIGURE 7.72 Phasor diagrams for the circuit in Figure 7.70. (a) $\omega^2 < 1/LC$. (b) Diagram at resonance. (c) $\omega^2 > 1/(LC)$.

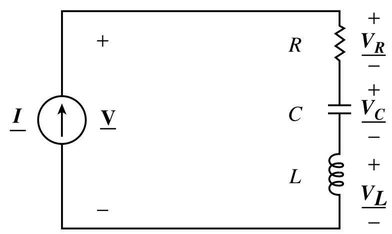


FIGURE 7.73 Series resonant circuit.

Another interesting aspect of the network in Figure 7.70 is that, at low frequencies ($\omega^2 < 1/(LC)$), $\underline{I}_1$ lags $\underline{V}$, and so the network appears inductive to the voltage source. At high frequencies ($\omega^2 > 1/(LC)$), $\underline{I}_1$ leads $\underline{V}$, and the network looks capacitive to the voltage source. At resonance, the network appears as only a resistor R_1 to the source. Figure 7.72 depicts phasor diagrams of $\underline{V}$ and $\underline{I}_1$ at low frequency, at resonance, and at high frequency.

Figure 7.73 is the second basic type of resonant circuit — a series resonant circuit which is excited by a sinusoidal current source with phasor transform $\underline{I} = I\ \angle 0°$. This circuit is dual to the circuit in Figure 7.70. The voltages across the network elements can be expressed as:

$$\underline{V_R} = IR;\ \underline{V_C} = -j\left(\frac{1}{\omega C}\right)I;\ \underline{V_L} = j\omega LI \tag{7.91}$$

Then, the voltage $\underline{V}$ is

$$\underline{V} = I\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] \tag{7.92}$$

The peak value of $\underline{V}$ is

$$V = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{7.93}$$

where I is the peak value of $\underline{I}$. The minimum value of V is

$$V_{\min} = IR \tag{7.94}$$

and this occurs at the frequency $\omega = 1/\sqrt{(LC)}$, which is the same resonance condition as for the circuit in Figure 7.70.

Equation (7.94) demonstrates that at resonance, the voltage across the LC subcircuit in Figure 7.73 is zero. However, the individual voltages across L and across C are not zero and can be quite large in magnitude depending on the values of the capacitor and inductor. These voltages are given by:

$$\underline{\boldsymbol{V}}_C = -jI\sqrt{\frac{L}{C}} \text{ and } \underline{\boldsymbol{V}}_L = jI\sqrt{\frac{L}{C}} \tag{7.95}$$

and therefore the voltage across the capacitor is exactly 180° out of phase with the voltage across the inductor.

At frequencies below resonance, $\underline{\boldsymbol{V}}$ lags $\underline{\boldsymbol{I}}$ in Figure 7.73, and therefore the circuit looks capacitive to the source. Above resonance, $\underline{\boldsymbol{V}}$ leads $\underline{\boldsymbol{I}}$, and the circuit looks inductive to the source. If the frequency of the source is $\omega = 1/\sqrt{(LC)}$ the circuit looks like a resistor of value R to the source.

Power in AC Circuits

If a sinusoidal voltage $v(t) = V \sin(\omega t + \theta_V)$ is applied to an LLFT network that possibly contains other sinusoidal sources having the same frequency ω, then a sinusoidal current $i(t) = I \sin(\omega t + \theta_1)$ flows at steady state as depicted in Figure 7.74. The instantaneous power delivered to the circuit by the voltage source is

$$p(t) = v(t)i(t) = VI\sin(\omega t + \theta_v)\sin(\omega t + \theta_I) \tag{7.96}$$

where the units of $p(t)$ are watts (W). With the aid of the trigonometric identity

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

we rewrite (7.96) as

$$p(t) = \frac{1}{2}VI\left[\cos(\theta_v - \theta_I) - \cos(2\omega t + \theta_v + \theta_I)\right] \tag{7.97}$$

The instantaneous power delivered to the network in Figure 7.74 has a component that is constant and another component that has a frequency twice that of the excitation. At different instances of time, $p(t)$ can be positive or negative, meaning that the voltage source is delivering power to the network or receiving power from the network, respectively.

In AC circuits, however, it is usually the average power P that is of more interest than the instantaneous power $p(t)$ because average power generates the heat or performs the work.

The average over a period of a periodic function $f(t)$ with period T is

$$\left[f(t)\right]_{\text{avg}} = F = \frac{1}{T}\int_0^T f(t)dt \tag{7.98}$$

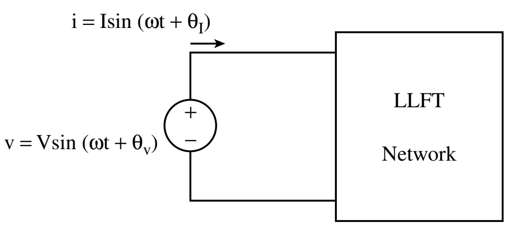


FIGURE 7.74 LLFT network that may contain other sinusoidal sources at the same frequency as the external generator.

The period of $p(t)$ in (7.97) is $T = \pi/\omega$, and so

$$[p(t)]_{avg} = P = \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} p(t)dt = \frac{1}{2}VI\cos(\theta_V - \theta_I) \tag{7.99}$$

The cosine term in (7.99) plays an important role in power calculations and so is designated as the Power Factor (*PF*). Thus,

$$\text{Power Factor} = PF = \cos(\theta_V - \theta_I) \tag{7.100}$$

If $|\theta_V - \theta_I| = \pi/2$, then $PF = 0$, and the average power delivered to the network in Figure 7.74 is zero; but if $PF = 1$, then P delivered to the network by the source is $VI/2$. If $0 < |\theta_v - \theta_I| < \pi/2$, then P is positive, and the source is delivering average power to the network. However, the network delivers average power to the source when P is negative, and this occurs if $\pi/2 < |\theta_v - \theta_I| < 3\pi/2$.

If the current leads the voltage in Figure 7.74, the convention is to consider *PF* as leading, and if current lags the voltage, the *PF* is regarded as lagging. However, it is not possible from *PF* alone to determine whether a current leads or lags voltage.

Example 24. Determine the average power delivered to the network shown in Figure 7.68(a).

Solution. The phasor transform of the applied voltage is $\underline{\boldsymbol{V}} = V\angle 0°$, and we determined in Example 23 that the current supplied was

$$\underline{\boldsymbol{I}}_1 = \frac{V\omega C e^{j\theta_1}}{\sqrt{\omega^2 C^2 R_1^2 + 1}}, \qquad \theta_1 = \tan^{-1}\frac{1}{\omega C R_1}$$

The power factor is

$$PF = \cos(0 - \theta_1) = \cos(\theta_1)$$

which, with the aid of the triangle in Figure 7.75, can be rewritten as

$$PF = \frac{\omega C R_1}{\sqrt{(\omega C R_1)^2 + 1}}$$

Thus, the average power delivered to the circuit is

$$P = \frac{1}{2}\frac{V^2\omega C}{\sqrt{(\omega C R_1)^2 + 1}}\left[\frac{\omega C R_1}{\sqrt{(\omega C R_1)^2 + 1}}\right] = \frac{V^2\omega^2 C^2 R_1}{2(\omega^2 C^2 R_1^2 + 1)} = \frac{I_1^2 R_1}{2}$$

□

We note that if R_1 were zero in the previous example, then $P = 0$ because the circuit would be purely capacitive, and *PF* would be zero.

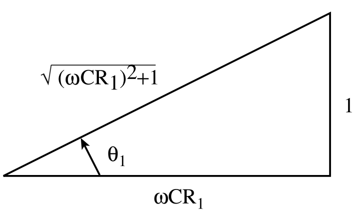


FIGURE 7.75 Triangle for determining *PF*.

If no sources are in the network in Figure 7.74, then the network terminal variables are related by:

$$\underline{\boldsymbol{V}} = \underline{\boldsymbol{I}}\, Z(j\omega) \tag{7.101}$$

where $Z(j\omega)$ is the input impedance of the network. Because Z is, general and complex, we can write it as:

$$Z(j\omega) = R(\omega) + jX(\omega) = |Z|e^{j\theta_Z} \tag{7.102}$$

where

$$R(\omega) = \mathcal{R}Z(j\omega); \quad X(\omega) = \mathcal{I}Z(j\omega)$$
$$\text{and } \theta_Z = \tan^{-1}\left(\frac{X(\omega)}{R(\omega)}\right) \tag{7.103}$$

In (7.103), the (real) function $X(\omega)$ is termed the reactance. Employing the polar form of the phasors, we can rewrite (7.101) as

$$V\angle\theta_V = I\angle\theta_I|Z|\angle\theta_Z = I|Z|\angle(\theta_I + \theta_Z) \tag{7.104}$$

Equating magnitudes and angles, we obtain

$$V = I|Z| \text{ and } \theta_V = \theta_I + \theta_Z \tag{7.105}$$

Thus, we can express P delivered to the network as

$$P = \frac{1}{2}VI\cos(\theta_V - \theta_I) = \frac{1}{2}I^2|Z|\cos\theta_Z \tag{7.106}$$

But $|Z|\cos(\theta_z) = R(\omega)$ so that

$$P = \frac{1}{2}I^2R(\omega) \tag{7.107}$$

Eq. (7.107) indicates that the real part of the impedance absorbs the power. The imaginary part of the impedance, $X(\omega)$, does not absorb average power. Example 24 in this section provides an illustration of (7.107).

An expression for average power in terms of the input admittance $Y(j\omega) = 1/Z(j\omega)$ can also be obtained. Again, if no sources are within the network, then the terminal variables in Figure 7.74 are related by

$$\underline{\boldsymbol{I}} = \underline{\boldsymbol{V}}\, Y(j\omega) \tag{7.108}$$

The admittance $Y(j\omega)$ can be written as

$$Y(j\omega) = |Y(j\omega)|e^{j\theta_Y} = G(\omega) + jB(\omega) \tag{7.109}$$

where $G(\omega)$ is conductance and $B(\omega)$ is susceptance, and where

$$G(\omega) = \mathcal{R}Y(j\omega); \quad B(\omega) = \mathcal{I}Y(j\omega)$$
$$\text{and } \theta_Y = \tan^{-1}\left[\frac{B(\omega)}{G(\omega)}\right] \tag{7.110}$$

Then, average power delivered to the network can be expressed as:

$$P = \frac{1}{2}V^2|Y|\cos\theta_Y = \frac{1}{2}V^2G(\omega) \tag{7.111}$$

If the network contains sinusoidal sources, then (7.99) should be employed to obtain P instead of (7.107) or (7.111).

Consider a resistor R with a voltage $v(t) = V\sin(\omega t)$ across it and therefore a current $i(t) = I\sin(\omega t) = v(t)/R$ through it. The instantaneous power dissipated by the resistor is

$$p(t) = v(t)i(t) = \frac{v^2(t)}{R} = i^2(t)R \tag{7.112}$$

The average power dissipated in R is

$$P = \frac{1}{T}\int_0^T i^2(t)\,Rdt = I_{eff}^2 R \tag{7.113}$$

where we have introduced the new constant I_{eff}. From (7.113), we can express I_{eff} as

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \tag{7.114}$$

This expression for I_{eff} can be read as "the square root of the mean (average) of the square of $i(t)$" or, more simply, as "the root mean square value of $i(t)$," or, even more succinctly, as "the *RMS* value of $i(t)$." Another designation for this constant is I_{rms}. Equation (7.114) can be extended to any periodic voltage or current.

The *RMS* value of a pure sine wave such as $i(t) = I\sin(\omega t + \theta_1)$ or $v(t) = V\sin(\omega t + \theta_v)$ is

$$I_{rms} = \frac{I}{\sqrt{2}} \text{ or } V_{rms} = \frac{V}{\sqrt{2}} \tag{7.115}$$

where I and V are the peak values of the sine waves. Normally, the voltages and currents listed on the nameplates of power equipment and household appliances are given in terms of *RMS* values instead of peak values. For example, a 120-V, 100-W lightbulb is expected to dissipate 100 W when a voltage $120(\sqrt{2})[\sin(\omega t)]$ is impressed across it. The peak value of this voltage is 170 V.

If we employ *RMS* values, (7.99) can be rewritten as

$$P = V_{rms}I_{rms}PF \tag{7.116}$$

Eq. (7.116) emphasizes the fact that the concept of *RMS* values of voltages and currents was developed in order to simplify the calculation of average power.

Because $PF = \cos(\theta_v - \theta_1)$, we can rewrite (7.116) as

$$\begin{aligned} P &= V_{rms}I_{rms}\cos(\theta_V - \theta_I) = \Re\left[V_{rms}e^{j\theta_V}I_{rms}e^{-j\theta_I}\right] \\ &= \Re\left[\underline{\boldsymbol{V}}\,\underline{\boldsymbol{I}}^*\right] \end{aligned} \tag{7.117}$$

where $\underline{\boldsymbol{I}}^*$ is the conjugate of $\boldsymbol{I}$. If $P = \Re\,[\underline{\boldsymbol{V}}\,\underline{\boldsymbol{I}}^*]$, the question arises as to what the imaginary part of $\underline{\boldsymbol{V}}\,\underline{\boldsymbol{I}}^*$ represents. This question leads naturally to the concept of complex power, denoted by the bold letter $\boldsymbol{S}$, which has the units of volt-amperes (VA). If P represents real power, then we can write

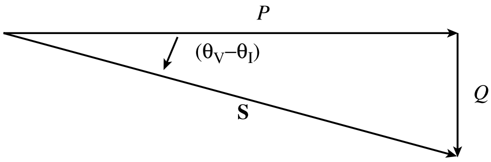


FIGURE 7.76 Power triangle for a capacitive circuit.

$$\boldsymbol{S} = P + jQ \tag{7.118}$$

where

$$\boldsymbol{S} = \underline{\boldsymbol{V}}\,\underline{\boldsymbol{I}}^* \tag{7.119}$$

and where

$$Q = \mathcal{I}\left[\underline{\boldsymbol{V}}\,\underline{\boldsymbol{I}}^*\right] = V_{rms} I_{rms} \sin(\theta_V - \theta_I) \tag{7.120}$$

Thus, Q represents imaginary or reactive power. The units of *Q* are VARs, which stands for volt-amperes reactive. Reactive power is not available for conversion into useful work. It is needed to establish and maintain the electric and magnetic fields associated with capacitors and inductors [4]. It is an overhead required for delivering *P* to loads, such as electric motors, that have a reactive part in their input impedances.

The components of complex power can be represented on a power triangle. Figure 7.76 is a power triangle for a capacitive circuit. Real and imaginary power are added as shown to yield the complex power **S**. Note that $(\theta_v - \theta_I)$ and *Q* are both negative for capacitive circuits. The following example illustrates the construction of a power triangle for an RL circuit.

Example 25. Determine the components of power delivered to the RL circuit in Figure 7.77. Provide a phasor diagram for the current and the voltages, construct a power triangle for the circuit, and show how the power diagram is related to the impedances of the circuit.

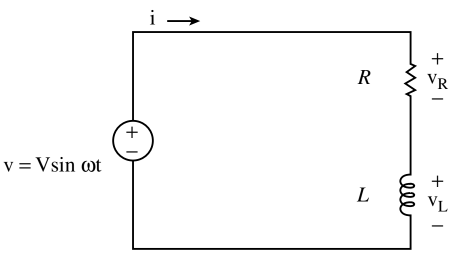


FIGURE 7.77 Network for Example 25.

Solution. We have

$$\underline{\boldsymbol{V}} = Ve^{j0} \text{ and } \underline{\boldsymbol{I}} = \frac{Ve^{j\theta_I}}{\sqrt{R^2 + (\omega L)^2}}$$

where

$$\theta_I = -\tan^{-1}\frac{\omega L}{R}$$

Because $\theta_v = 0$, PF is

$$PF = \cos(\theta_V - \theta_I) = \frac{R}{\sqrt{R^2 + (\omega L)^2}}$$

and is lagging. The voltages across R and L are given by:

$$\underline{\boldsymbol{V}_R} = \underline{\boldsymbol{I}}R = \frac{V\,\mathrm{R}e^{j\theta_I}}{\sqrt{R^2 + (\omega L)^2}}$$

$$\underline{\boldsymbol{V}_L} = j\omega L\,\underline{\boldsymbol{I}} = \frac{V\omega L}{\sqrt{R^2 + (\omega L)^2}}e^{j\left(\frac{\pi}{2}+\theta_I\right)}$$

and Z is

$$Z = R + j\omega L = \sqrt{R^2 + (\omega L)^2}\,e^{-j\theta_I}$$

The real and imaginary components of the complex power are simply calculated as:

$$P = I_{rms}^2 R(\omega) = \frac{V_{rms}^2 R}{R^2 + (\omega L)^2}$$

$$Q = \frac{V_{rms}^2 \omega L}{R^2 + (\omega L)^2}$$

Figure 7.78 presents the phasor diagram for this circuit in which we have taken the reference phasor as $\underline{\boldsymbol{I}}$ and therefore have shown $\underline{\boldsymbol{V}}$ leading $\underline{\boldsymbol{I}}$ by $(\theta_v - \theta_1)$. Also, we have moved $\underline{\boldsymbol{V}}_L$ parallel to itself to form a triangle. These operations cause the phasor diagram to be similar to the power triangle. Figure 7.79(a) shows a representation for the impedance in Figure 7.77. If each side of the triangle in Figure 7.79(a) is multiplied by I_{rms}, then we obtain voltage triangle in Figure 7.79(b). Next, we multiply the sides of the voltage triangle by I_{rms} again to obtain the power triangle in Figure 7.79(c). The horizontal side is the average power P, the vertical side is Q, and the hypotenuse has a length that represents the magnitude of the complex power $\boldsymbol{S}$. All three triangles in Figure 7.79 are similar. The angles between sides are preserved. □

If P remains constant in Figure 7.76, but the magnitude of the angle becomes larger so that the magnitude of Q increases, then [**S**] increases. If the magnitude of the voltage is fixed, then the magnitude of the current supplied must increase. But then, either power would be lost in the form of heat in the wires supplying the load or larger diameter, more expensive wires, would be needed. For this reason, power companies that supply power to large manufacturing firms that have many large motors impose unfavorable rates. However, the manufacturing firm can improve its rates if it improves its power factor. The following example illustrates how improving (correcting) PF is done.

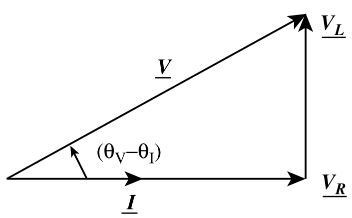


FIGURE 7.78 Phasor diagram for Example 25.

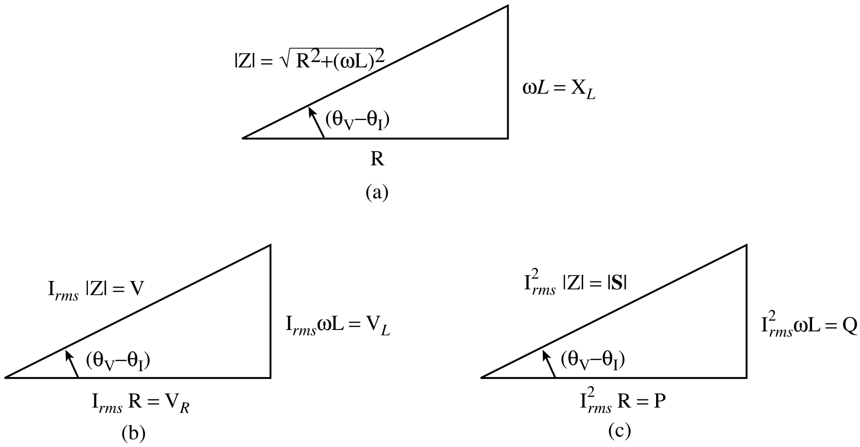


FIGURE 7.79 (a) Impedance triangle for circuit in Example 25. (b) Corresponding voltage triangle. (c) Power triangle.

Example 26. Determine the value of the capacitor to be connected in parallel with the RL circuit in Figure 7.80 to improve the *PF* of the overall circuit to one. The excitation is a voltage source having an amplitude of 120 V *RMS* and frequency 2π(60 Hz) = 377 rad/s. What are the *RMS* values of the current supplied by the source at steady state before and after the capacitor is connected?

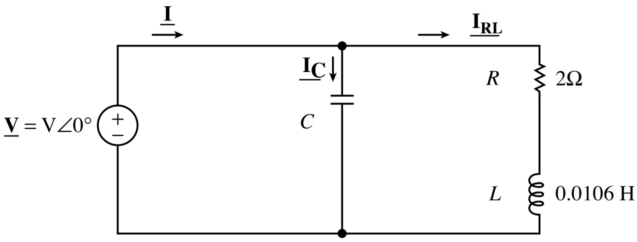


FIGURE 7.80 Circuit for Example 26.

Solution. The current through the RL branch in Figure 7.80 is

$$\underline{I_{RL}} = \frac{Ve^{j\theta}}{\sqrt{R^2 + (\omega L)^2}}; \quad \theta = -\tan^{-1}\frac{\omega L}{R}$$

and the current through the capacitor is

$$\underline{I_C} = j\omega CV = V\omega Ce^{j(\pi/2)}$$

Thus, the current supplied by the source to the *RLC* network is

$$\begin{aligned}\underline{I} &= \underline{I_{RL}} + \underline{I_C}\\ &= \frac{V\cos\theta}{\sqrt{R^2+(\omega L)^2}} + jV\left[\frac{-\omega L}{R^2+(\omega L)^2} + \omega C\right]\end{aligned}$$

To improve the *PF* to one, the current $\underline{I}$ should be in phase with $\underline{V}$. Thus, we set the imaginary term in the equation for $\underline{I}$ equal to zero, yielding:

$$C = \frac{L}{R^2 + (\omega L)^2} = 530\ \mu F$$

a rather large capacitor. Before this capacitor is connected, the *RMS* value of the current supplied by the voltage source is I_{rms} = 26.833 amps. After the capacitor is connected, the source has to supply only 12 amps *RMS*, a considerable reduction. In both cases, *P* delivered to the load is the same.

The following example also illustrates *PF* improvement.

Example 27. A load with *PF* = 0.7 lagging, depicted in Figure 7.81, consumes 12 kW of power. The line voltage supplied is 220 V *RMS* at 60 Hz. Find the size of the capacitor needed to correct the *PF* to 0.9 lagging, and determine the values of the currents supplied by the source both before and after the *PF* is corrected.

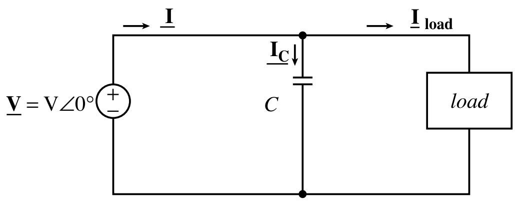


FIGURE 7.81 Circuit for Example 27 showing the load and the capacitor to be connected in parallel with the load to improve the power factor.

Solution. We will take the phase of the line voltage to be 0°. From $P = V_{rms} I_{rms}\, PF$ = 12 kW, we obtain I_{rms} = 77.922 amps. Because *PF* is 0.7 lagging, the phase of the current through the load relative to the phase of the line voltage is $-\cos^{-1}(0.7) = -45.57°$. Therefore, $\underline{I}_{load} = 77.922\angle(-45.57°)$ amps *RMS*. When *C* is connected in parallel with the load,

$$\begin{aligned} \underline{I} = \underline{I}_C + \underline{I}_{load} &= 220(377)jC + 77.922e^{-j0.7954} \\ &= 54.54 - j[55.64 - 82{,}940C] \end{aligned}$$

If the *PF* were to be corrected to unity, we would set the imaginary part of the previous expression for current to zero; but this would require a larger capacitor (671 μ*F*), which may be uneconomical. Instead, to retain a lagging but improved *PF* = 0.9, and corresponding to the current lagging the voltage by 25.84°, we write

$$0.9 = \frac{54.54}{\sqrt{54.54^2 + (55.64 - 82{,}940C)^2}}$$

Therefore, *C* = 352 μ*F*. The line current is now

$$\underline{I} = \underline{I}_C + \underline{I}_{load} = 60.615\angle(-25.87°)\text{amps } RMS$$

□

Previous examples have employed ideal voltage sources to supply power to networks. However, in many electronic applications, the source has a fixed impedance associated with it, and the problem is to

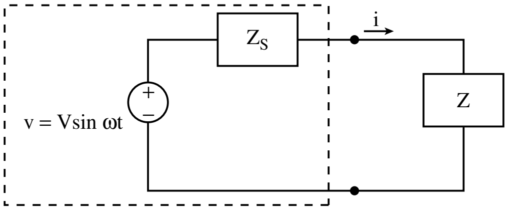


FIGURE 7.82 Z_s is fixed, and Z is to be chosen so that maximum average power is transferred to Z.

obtain the maximum average power transferred to the load [2]. Here, we assume that the resistance and reactance of the load can be independently adjusted. Let the source impedance be:

$$Z_S(j\omega) = R_S(\omega) + jX_S(\omega)$$

The load impedance is denoted as

$$Z(j\omega) = R(\omega) + jX(\omega)$$

Figure 7.82 depicts these impedances. We assume that all the elements, including the voltage source, within the box formed by the dotted lines are fixed. The voltage source is $v(t) = V \sin(\omega t)$, and thus $i(t) = I \sin(\omega t+\theta)$, where V and I are peak values and

$$\theta = -\tan^{-1}\left[\frac{X_S(\omega) + X(\omega)}{R_S(\omega) + R(\omega)}\right]$$

The average power delivered to Z is

$$P = I_{rms}^2 R(\omega)$$

where $I_{rms} = I/\sqrt{2}$ and

$$I = \frac{V}{\sqrt{\left[R_S(\omega) + R(\omega)\right]^2 + \left[X_S(\omega) + X(\omega)\right]^2}} \tag{7.121}$$

Thus, the average power delivered to Z can be written as

$$P = \frac{V_{rms}^2 R(\omega)}{\left[R_S(\omega) + R(\omega)\right]^2 + \left[X_S(\omega) + X(\omega)\right]^2} \tag{7.122}$$

To maximize P, we first note that the term $[X_s(\omega) + X(\omega)]^2$ is always positive, and so this term always contributes to a larger denominator unless it is zero. Thus, we set

$$X(\omega) = -X_S(\omega) \tag{7.123}$$

and (7.122) becomes

$$P = \frac{V_{rms}^2 R(\omega)}{\left[R_S(\omega) + R(\omega)\right]^2} \tag{7.124}$$

Second, we set the partial derivative with respect to $R(\omega)$ of the expression in (7.124) to zero to obtain

$$\frac{\partial P}{\partial R} = V_{rms}^2 \frac{(R_S + R)^2 - 2R(R_S + R)}{(R_S + R)^4} = 0 \tag{7.125}$$

Eq. (7.125) is satisfied for

$$R(\omega) = R_S(\omega) \tag{7.126}$$

and this value of $R(\omega)$ together with $X(\omega) = -X_s(\omega)$, yields maximum average power transferred to Z. Thus, we should adjust Z to:

$$Z(j\omega) = Z_S^*(j\omega) \tag{7.127}$$

and we obtain

$$P_{max} = \frac{V_{rms}^2}{4R(\omega)} \tag{7.128}$$

Example 28. Find Z for the network in Figure 7.83 so that maximum average power is transferred to Z. Determine the value of P_{max}.

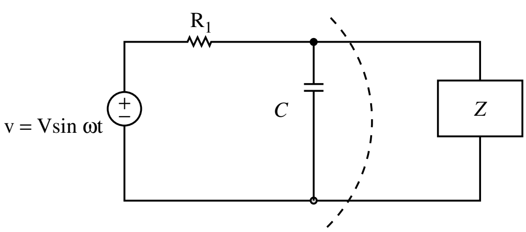


FIGURE 7.83 Circuit for Example 28.

Solution. We first obtain the Thévenin equivalent of the circuit to the left of the dotted arc in Figure 7.83 in order to reduce the circuit to the form of Figure 7.82.

$$\underline{\boldsymbol{V}_{TH}} = \frac{\underline{\boldsymbol{V}}}{1 + j\omega R_1 C}$$

$$Z_{TH} = \frac{R_1}{1 + j\omega R_1 C}$$

Thus,

$$Z = Z_{TH}^* = \frac{R_1}{1 - j\omega R_1 C} = \frac{R_1}{1 + \dfrac{\omega R_1 C}{j}}$$

$$= \frac{jR_1}{j + \omega R_1 C} = \frac{j\dfrac{1}{\omega C}R_1}{R_1 + j\dfrac{1}{\omega C}}$$

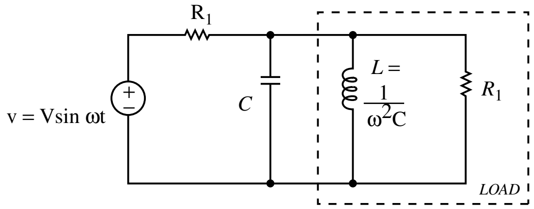


FIGURE 7.84 Circuit with load chosen to obtain maximum average power.

The term $j(\omega C)$ appears inductive (at a single frequency), and so we equate it to $j\omega L$ to obtain:

$$L = \frac{1}{\omega^2 C}$$

The impedance Z is therefore formed by the parallel connection of a resistor R_1 with the inductor L. Figure 7.84 depicts the resulting circuit. To determine P_{max}, we note that the capacitor and inductor constitute a parallel circuit which is resonant at the frequency of excitation. It therefore appears as an open circuit to the source. Thus, P_{max} is easily obtained as:

$$P_{\max} = I_{rms}^2 R_1 = \frac{V^2}{8R_1}$$

where V is the peak value of $v(t)$. □

Suppose Z is fixed and Z_s is adjustable in Figure 7.82. What should Z_s be so that maximum average power is delivered to Z? This is a problem that is applicable in the design of electronic amplifiers. The average power delivered to Z is given by (7.122), and to maximize P, we set $X_s(\omega) = -X(\omega)$ as before. We therefore obtain (7.124) again; but if R_s is adjustable instead of R, we see from (7.124) that P_{max} is obtained when R_s equals zero.

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