9

Analysis in the Time Domain

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9.1 Signal Types

Introduction

Because information into and out of a circuit is carried via time domain signals we look first at some of the basic signals used in continuous time circuits. All signals are taken to depend on continuous time t over the full range $-\infty < t < \infty$. It is important to realize that not all signals of interest are functions in the strict mathematical sense; we must go beyond them to generalized functions (e.g., the impulse), which play a very important part in the signal processing theory of circuits.

Step, Impulse, and Ramp

The unit **step** function, denoted $1(\cdot)$, characterizes sudden jumps, such as when a signal is turned on or a switch is thrown; it can be used to form pulses, to select portions of other functions, and to define the ramp and impulse as its integral and derivative. The unit step function is discontinuous and jumps between two values, 0 and 1, with the time of jump between the two taken as $t = 0$. Precisely,

$$1(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{9.1}$$

which is illustrated in Figure 9.1 along with some of the functions to follow.

Here, the value at the jump point, $t = 0$, purposely has been left free because normally it is immaterial and specifying it can lead to paradoxical results. Physical step functions used in the laboratory are actually continuous functions that have a continuous rise between 0 and 1, which occurs over a very short time. Nevertheless, instances occur in which one may wish to set $1(0)$ equal to 0 or to 1 or to 1/2 (the latter, for example, when calculating the values of a Fourier series at a discontinuity). By shifting the time

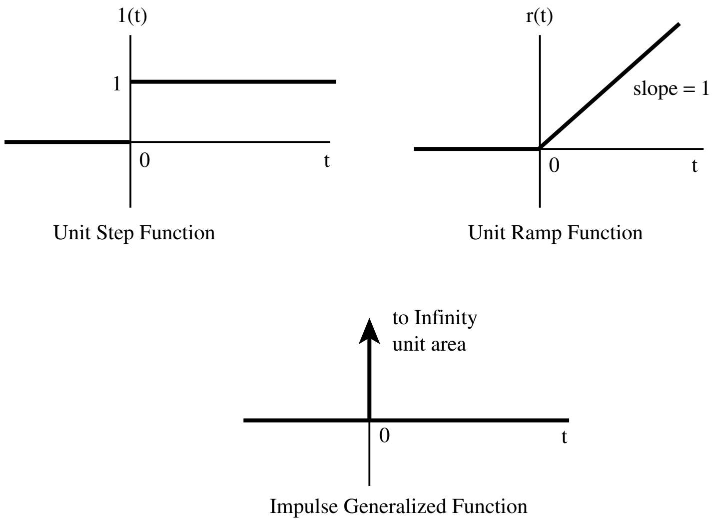

FIGURE 9.1 Step, ramp, and impulse functions.

argument the jump can be made to occur at any time, and by multiplying by a factor the height can be changed. For example, $1(t - t_0)$ has a jump at time t_0 and $a[1(t) - 1(t - t_0)]$ is a pulse of width t_0 and height a going up to a at $t = 0$ and down to 0 at time t_0. If $a = a(t)$ is a function of time, then that portion of $a(t)$ between 0 and t_0 is selected. The unit **ramp**, $r(\cdot)$ is the continuous function which ramps up linearly (with unit slope) from zero starting at $t = 0$; the ramp results from the unit step by integration

$$r(t) = \int_{-\infty}^{t} 1(\tau)d\tau = t1(t) = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if t} < 0 \end{cases} \tag{9.2}$$

As a consequence the unit step is the derivative of the unit ramp, while differentiating the unit step yields the unit **impulse** generalized function, $\delta(\cdot)$ that is

$$\delta(t) = \frac{d1(t)}{dt} = \frac{d^2r(t)}{dt^2} \tag{9.3}$$

In other words, the unit impulse is such that its integral is the unit step; that is, its area at the origin, $t = 0$, is 1. The impulse acts to sample continuous functions which multiply it, i.e.,

$$a(t)\delta(t - t_0) = a(t_0)\delta(t - t_0) \tag{9.4}$$

This sampling property yields an important integral representation of a signal $x(\cdot)$

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t)\delta(t - \tau)d\tau = x(t) \int_{-\infty}^{\infty} \delta(t - \tau)d\tau \end{aligned} \tag{9.5}$$

where the validity of the first line is seen from the second line, and the fact that the integral of the impulse through its jump point is unity. Equation (9.5) is actually valid even when $x(\cdot)$ is discontinuous and, consequently, is a fundamental equation for linear circuit theory. Differentiating $\delta(t)$ yields an even more discontinuous object, the doublet $\delta'(\cdot)$. Strictly speaking, the impulse, all its derivatives, and signals of that class are not functions in the classical sense, but rather they are operators [1] or functionals [2], called generalized functions or, often, distributions. Their evaluations take place via test functions, just as voltages are evaluated on test meters.

The importance of the impulse lies in the fact that if a linear time-invariant system is excited by the unit impulse, then the response, naturally called the impulse response, is the inverse Laplace transform of the network function. In fact, if $h(t)$ is the impulse response of a linear time-invariant (continuous and continuous time) circuit, the forced response $y(t)$ to any input $u(t)$ can be obtained without leaving the time domain by use of the convolution integral, with the operation of convolution denoted by *,

$$y(t) = h * u = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \tag{9.6}$$

Equation (9.6) is mathematically rigorous, but justified on physical grounds through (9.5) as follows. If we let $h(t)$ be the output when $\delta(t)$ is the input, then, by time invariance, $h(t-\tau)$ is the output when the input is shifted to $\delta(t-\tau)$. Scaling the latter by $u(\tau)$ and summing via the integral, as designated in (9.5), we obtain a representation of the input $u(t)$. This must result in the output representation being in the form of (9.6) by linearity of the system through similar scaling and summing of $h(t-\tau)$, as was performed on the input.

Sinusoids

Sinusoidal signals are important because they are self-reproducing functions (i.e., eigenfunctions) of linear time-invariant circuits. This is true basically because the derivatives of sinusoids are sinusoidal. As such, sinusoids are also the natural outputs of oscillators and are delivered in power sources, including laboratory signal generators and electricity for the home derived from the power company.

Eternal

Eternal signals are defined as being of the same nature for all time, $-\infty < t < \infty$, in which case an eternal cosine repeats itself eternally in both directions of time, with an origin of time, $t = 0$, being arbitrarily fixed. Because eternal sinusoids have been turned on forever, they are useful in describing the steady operation of circuits. In particular, the signal $A\cos(\omega t + \theta)$ over $-\infty < t < \infty$ defines an eternal cosine of amplitude A, radian frequency $\omega = 2\pi f$ (with f being real frequency, in Hertz, which are cycles per second), at phase angle θ (in radians and with respect to the origin of time), with A, ω, and θ real numbers. When $\theta = \pi/2$ this cosine also represents a sine, so that all eternal sinusoidal signals are contained in the expression $A\cos(\omega t + \theta)$.

At times, it is important to work with sinusoids that have an exponential envelope, with the possibility that the envelope increases or decreases with time, that is, with positively or negatively damped sinusoids. These are described by $Ae^{st}\cos(\omega t + \theta)$, where the real number is the damping factor, giving signals that damp out in time when the damping factor is positive and signals that increase with time when the damping factor is negative. Of most importance when working with this class of signals is the identity

$$e^{\sigma t + j\omega t} = e^{st} = e^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right] \tag{9.7}$$

where $s = \sigma + j\omega$ with $j = \sqrt{-1}$. Here, s is called the **complex frequency**, with its imaginary part being the real (radian) frequency, ω. When no damping is present, $s = j\omega$, in which case the exponential form of (9.7) represents pure sinusoids. In fact, we see in this expression that the cosine is the real part of an exponential and the sine is its imaginary part. Because exponentials are usually easier than sinusoids to treat analytically, the consequence for real linear networks is that we can do most of the calculations with exponentials and convert back to sinusoids at the end. In other words, if a real linear system has a cosine or a damped cosine as a true input, it can be analyzed by using instead the exponential of which it is the real part as its (fictitious) input, finding the resulting (fictitious) exponential output, and then taking the real part at the end of the calculations to obtain the true output for the true input. Because exponentials are probably the easiest signals to work with in theory, the use of exponentials rather than sinusoids usually greatly simplifies the theory and calculations for circuits operating under steady-state conditions.

Causal

Because practical circuits have not existed since $t = -\infty$ they usually begin to be considered at a suitable starting time, taken to be $t = 0$, in which case the associated signals can be considered to be zero for $t < 0$. Mathematically, these functions are said to have support bounded on the left. The support of a signal is (the closure of) that set of times for which the signal is non-zero, therefore, the support of these signals is bounded on the left by zero. When signals are discontinuous functions they have the important property that they can be represented by multiplying with unit step functions signals which are differentiable and have nonbounded support. For example, $g(t) = e^{st} \cdot 1(t)$ has a jump at $t = 0$ with support at the half line 0 to ∞ but has e^{st} infinitely differential of "eternal" support.

A **causal** circuit is one for which the response is only nonzero after the input becomes nonzero. Thus, if the inputs are zero for $t < 0$, the outputs of causal circuits are also zero for $t < 0$. In such cases the impulse response, $h(t)$, or the response to an input impulse of "infinite jump" at $t = 0$, satisfies $h(t) = 0$ for $t < 0$ and the convolution form of the output, (9.4), takes the form

$$y(t) = \left[\int_0^t h(t-\tau)u(\tau)d\tau\right]1(t) \tag{9.8}$$

Periodic and Aperiodic Waveforms

The pure sinusoids, although not the sinusoids with nonzero damping, are special cases of periodic signals. In other words, ones which repeat themselves in time every T seconds, where T is the period. Precisely, a time-domain signal $g(\cdot)$ is **periodic** of period T if $g(t) = g(t + T)$, where normally T is taken to be the smallest nonzero T for which this is true. In the case of the sinusoids, $A\cos(\omega t + \theta)$ with $\omega = 2\pi f$, the period is given by $T = 1/f$ because $\{2\pi[f(t + T)] + \theta\} = \{2\pi ft + 2\pi(fT) + \theta\} = \{2\pi ft + (2\pi + \theta)\}$, and sinusoids are unchanged by a change of 2π in the phase angle. Periodic signals need to be specified over only one period of time, e.g., $0 \le t < T$, and then can be extended periodically for all time by using $t = t \bmod(T)$ where $\bmod(\cdot)$ is the modulus function; in other words, periodic signals can be looked upon as being defined on a circle, if we imagine the circle as being a clock face.

Periodic signals represent rhythms of a system and, as such, contain recurring information. As many phycial systems, especially biomedical systems, either possess directly or to a very good approximation such rhythms, the periodic signals are of considerable importance. Even though countless periodic signals are available besides the sinusoids, it is important to note that almost all can be represented by a Fourier series. Exponentials are eigenfunctions for linear circuits, thus, the Fourier series is most conveiently expressed for circuit considerations in terms of the exponential form. If $g(t) = g(t + T)$, then

$$g(t) \cong \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi nt/T)} \tag{9.9}$$

where the coefficients are complex and are given by

$$c_n = \frac{1}{T}\int_0^T g(t)e^{-j(2\pi nt/T)}dt = a_n + jb_n \tag{9.10}$$

Strictly speaking, the integral is over the half-open interval $[0,T)$ as seen by considering $g(\cdot)$ defined on the circle. In (9.9), the symbol $\cong$ is used to designate the expression on the right as a representation that may not exactly agree numerically with the left side at every point when $g(\cdot)$ is a function; for example, at discontinuities the average is obtained on the right side. If $g(\cdot)$ is real, that is, $g(t) = g(t)^*$, where the superscript * denotes complex conjugate, then the complex coefficients c_n satisfy $c_n = c_{-n}^*$. In this case the real coefficients a_n and b_n in (9.10) are even and odd in the indices; n and the a_n combine to give a series in terms of cosines, and the b_n gives a series in terms of sines.

As an example the square wave, sqw(t), can be defined by

$$\text{sqw}(t) = 1(t) - 1\left(t - [T/2]\right) \quad 0 \le t < T \tag{9.11}$$

and then extended periodically to $-\infty < t < \infty$ by taking $t = t\text{mod}(T)$. The exponential Fourier series coefficients are readily found from (9.10) to be

$$c_n = \begin{cases} 1/2 & \text{if } n = 0 \\ \dfrac{1}{j\pi n}\begin{cases} 0 & \text{if } n = 2k \neq 0 \,(\text{even} \neq 0) \\ 1 & \text{if } n = 2k+1 \,(\text{odd}) \end{cases} \end{cases} \tag{9.12}$$

for which the Fourier series is

$$\text{sqw}(t) \cong \frac{1}{2} + \sum_{k=-\infty}^{\infty} \frac{1}{j\pi[2k+1]} e^{j2\pi[2k+1]t/T} \tag{9.13}$$

The derivative of sqw(t) is a periodic set of impulses

$$\frac{d[\text{sqw}(t)]}{dt} = \delta(t) - \delta\left(t - [T/2]\right) \quad 0 \le t < T \tag{9.13}$$

for which the exponential Fourier series is easily found by differentiating (9.13), or by direct calculation from (9.10), to be

$$\sum_{i=-\infty}^{\infty}\left(\delta(t - iT) - \delta\left(t - iT - [T/2]\right)\right) \cong \sum_{k=-\infty}^{\infty} \frac{2}{T} e^{j(2\pi[2k+1]t/T)} \tag{9.15}$$

Combining the exponentials allows for a sine representation of the periodic generalized function signal. Further differentiation can take place, and by integrating (9.15) we get the Fourier series for the square wave if the appropriate constant of integration is added to give the DC value of the signal. Likewise, a further integration will yield the Fourier series for the sawtooth periodic signal, and so on.

The importance of these Fourier series representations is that a circuit having periodic signals can always be considered to be processing these signals as exponential signals, which are usually self-reproducing signals for the system, making the design or analysis easy. The Fourier series also allows visualization of which radian frequencies, $2\pi n/T$, may be important to filter out or emphasize. In many common cases, especially for periodically pulsed circuits, the series may be expressed in terms of impulses. Thus, the impulse response of the circuit can be used in conjunction with the Fourier series.

References

[1] J. Mikusinski, *Operational Calculus*, 2nd ed., New York; Pergamon Press, 1983.
[2] A. Zemanian, *Distribution Theory and Transform Analysis*, New York: McGraw-Hill, 1965.

9.2 First-Order Circuits

Introduction

First-order circuits are fundamental to the design of circuits because higher order circuits can be considered to be constructed of them. Here, we limit ourselves to single-input-output linear time-invariant circuits for which we take the definition of a first-order circuit to be one described by the differential equation

$$d_1 \cdot \frac{dy}{dt} + d_0 \cdot y = n_1 \cdot \frac{du}{dt} + n_0 \cdot u \tag{9.16}$$

where d_0 and d_1 are "denominator" constants and n_0 and n_1 are "numerator" constants, $y = y(\cdot)$ is the output and $u = u(\cdot)$ is the input, and both u and y are generalized functions of time t. So that the circuit truly will be first order, we require that $d_1 \cdot n_0 - d_0 \cdot n_1 \neq 0$, which guarantees that at least one of the derivatives is actually present, but if both derivatives occur, the expressions in y and in u are not proportional, which would lead to cancellation, forcing y and u to be constant multiples of each other. Because a factorization of real higher-order systems may lead to complex first-order systems, we will allow the numerator and denominator constants to be complex numbers; thus, y and u may be complex-valued functions.

If the derivative is treated as an operator, $p = d[\cdot]/dt$, then (9.16) can be conveniently written as

$$y = \frac{n_1 p + n_0}{d_1 p + d_0} u = \begin{cases} \left[\dfrac{n_1}{d_0} p + \dfrac{n_0}{d_0}\right] u & \text{if } d_1 = 0 \\ \left[\dfrac{n_1}{d_1} + \dfrac{d_1 n_0 - d_0 n_1}{p + (d_0/d_1)}\right] u & \text{if } d_1 \neq 0 \end{cases} \tag{9.17}$$

where the two cases in terms of d_1 are of interest because they provide different forms of responses, each of which frequently occurs in first-order circuits. As indicated by (9.17), the transfer function

$$H(p) = \frac{n_1 p + n_0}{d_1 p + d_0} \tag{9.18}$$

is an operator (as a function of the derivative operator p), which characterizes the circuit. Table 9.1 lists some of the more important types of different first-order circuits along with their transfer functions and causal impulse responses.

The following treatment somewhat follows that given in [1], although with a slightly different orientation in order to handle all linear time-invariant continuous time continuous circuits.

Zero-Input and Zero-State Response

The response of a linear circuit is, via the linearity, the sum of two responses, one due to the input when the circuit is initially in the zero state, called the **zero-state response**, and the other due to the initial state when no input is present, the **zero-input response**. By the linearity the total response is the sum of the two separate responses, and thus we may proceed to find each separately. In order to investigate these two types of responses, we introduce the state vector $x(\cdot)$ and the state-space representation (as previously $p = d[\cdot]/dt$)

$$\begin{aligned} px &= Ax + Bu \\ y &= Cx + Du + Epu \end{aligned} \tag{9.19}$$

TABLE 9.1 Typical Transfer Functions of First-Order Circuits

Transfer Function	Description	Impulse Response
$\frac{n_1}{d_0}p$	Differentiator	$\frac{n_1}{d_0}\delta'(t)$
$\frac{n_0}{d_1 p}$	Integrator	$\frac{n_0}{d_1}1(t)$
$\frac{n_1 p + n_0}{d_1}$	Leaky differentiator	$\frac{n_0}{d_1}\delta(t) + \frac{n_1}{d_1}\delta'(t)$
$\frac{n_0}{d_1 p + d_0}$	Low-pass filter; lossy integrator	$\frac{n_0}{d_1}e^{-\frac{d_0}{d_1}t}\cdot 1(t)$
$\frac{n_1 p}{d_1 p + d_0}$	High-pass filter	$\frac{n_1}{d_1}\delta(t) + \frac{n_1 d_0}{d_1^2}e^{-\frac{d_0}{d_1}t}\cdot 1(t)$
$\frac{n_1}{d_1}\frac{p-(d_0/d_1)}{p+(d_0/d_1)}$	All-pass filter	$\frac{n_1}{d_1}\left[\delta(t) - 2\frac{d_0}{d_1}e^{-\frac{d_0}{d_1}t}\cdot 1(t)\right]$

where A, B, C, D, E are constant matrices. For our first-order circuit two cases are exhibited, depending upon d_1 being zero or not. In the case of $d_1 = 0$,

$$y = (n_1/d_0)u + (n_1/d_0)pu \qquad d_1 = 0 \tag{9.20a}$$

Here, $C = 0$ and A and B can be chosen anything, including empty. When $d_1 \neq 0$, our first-order circuit has the following set of (minimal size) state-variable equations

$$\begin{aligned} px &= \left[-\frac{d_0}{d_1}\right]\cdot x + \left[d_1 n_0 - d_0 n_1\right]\cdot u \\ y &= [1]\cdot x + \left[\frac{n_1}{d_1}\right]\cdot u \end{aligned} \qquad d_1 \neq 0 \tag{9.20b}$$

By choosing $u = 0$ in (9.2), we obtain the equations that yield the zero input response. Specifically, the zero-input response is

$$y(t) = \begin{cases} 0 & \text{if } d_1 = 0 \\ e^{-\frac{d_0}{d_1}t}\cdot y(0) & \text{if } d_1 \neq 0 \end{cases} \tag{9.21}$$

which is also true by direct substitution into (9.16). Here, we have set, in the $d_1 \neq 0$ case, the initial value of the state, $x(0)$, equal to the initial value of the output, $y(0)$, which is valid by our choice of state-space equations. Note that (9.21) is valid for all time and y at $t = 0$ assumes the assigned initial value $y(0)$, which must be zero when the input is zero and no derivative occurs on the output.

The zero-state response is explained as the solution of (9.21) when $x(0) = 0$. In the case that $d_1 = 0$, the zero-state response is

$$y = \frac{n_0}{d_0}u + \frac{n_1}{d_0}pu = \left\{\frac{n_0}{d_0}\delta(t) + \frac{n_1}{d_0}\delta'(t)\right\} * u \qquad d_1 = 0 \tag{9.22a}$$

where $*$ denotes convolution, $\delta(\cdot)$ is the unit impulse, and $1(\cdot)$ is the unit step function. While in the case that $d_1 \neq 0$

$$y = \left\{ \frac{n_1}{d_1} \delta(t) + \left[\frac{d_1 n_0 - d_0 n_1}{d_1} \right] e^{-\frac{d_0}{d_1} t} 1(t) \right\} * u \qquad d_1 \neq 0 \tag{9.22b}$$

which is found by eliminating x from (9.20b) and can be checked by direct substitution into (9.16). The terms in the braces are the causal impulse responses, $h(t)$, which are checked by letting $u = \delta$ with otherwise zero initial conditions, that is, with the circuit initially in the zero state. Actually, infinitely many noncausal impulse responses could be used in (9.22b). One such response is found by replacing $1(t)$ by $-1(-t)$]. However, physically the causal responses are of most interest.

If $d_1 \neq 0$, the form of the responses is determined by the constant d_0/d_1, the reciprocal of which (when $d_0 \neq 0$) is called the **time constant**, t_c, of the circuit because the circuit impulse response decays to $1/e$ at time $t_c = d_1/d_0$. If the time constant is positive, the zero-input and the impulse responses asymptotically decay to zero as time approaches positive infinity, and the circuit is said to be **asymptotically stable**. On the other hand, if the time constant is negative, then these two responses grow without bounds as time approaches plus infinity, and the circuit is called unstable. It should be noted that as time goes in the reverse direction to minus infinity, the unstable zero-input response decays to zero. If $d_0/d_1 = 0$ the zero-input and impulse responses are still stable, but neither decay nor grow as time increases beyond zero.

By linearity of the circuit and its state-space equations, the total response is the sum of the zero-state response and the zero-input response; thus, even when $d_0 = 0$ or $d_1 = 0$

$$y(t) = e^{-\frac{d_0}{d_1}} y_0 + h(t) * u(t) \tag{9.23}$$

Assuming that u and h are zero for $t < 0$ their convolution is also zero for $t < 0$, although not necessarily at $t = 0$, where it may even take on impulsive behavior. In such a case, we see that y_0 is the value of the output instantaneously before $t = 0$. If we are interested only in the circuit for $t > 0$, surprisingly, an input will yield the zero input response. That is, an equivalent input u_0 exists, which will yield the zero input response for $t > 0$, this being $u_0(t) = d_1 y_0 \exp(-td_0/d_1)1(t)$. Thus, $y = h * (u + u_0)$ gives the same result as (9.23).

When $d_1 = 0$, the circuit acts as a differentiator and within the state-space framework it is treated as a special case. However, in practice it is not a special case because the current, i, versus voltage, v, for a capacitor of capacitance C, in parallel with a resistor of conductance G is described by $i = Cpv + Gv$. Consequently, it is worth noting that all cases can be handled identically in the semistate description

$$\begin{bmatrix} d_1 & d_1 - 1 \\ 0 & 0 \end{bmatrix} px = \begin{bmatrix} -d_0 & -d_0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} n_0 \\ n_1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \tag{9.24}$$

where $x(\cdot)$ is the semistate instead of the state, although the first components of the two vectors agree in many cases. In other words, the semistate description is more general than the state description, and handles all circuits in a more convenient fashion [2].

Transient and Steady-State Responses

This section considers stable circuits, although the techniques are developed so that they apply to other situations. In the asymptotically stable case, the zero input response decays eventually to zero; that is, transient responses due to initial conditions eventually will not be felt and concentration can be placed upon the zero-state response. Considering first eternal exponential inputs, $u(t) = U \exp(st)$ for $-\infty < t < \infty$ at the complex frequency $s = \sigma + j\omega$, where s is chosen as different from the natural frequency $s_n = -d_0/d_1 = -1/t_c$ and U is a constant, we note that the response is $y(t) = Y(s) \exp(st)$, as is observed by direct substitution into (9.16); this substitution yields directly

$$Y(s) = \frac{n_1 s + n_0}{d_1 s + d_0} \cdot U \tag{9.25}$$

where $y(t) = Y(s) \exp(st)$ for $u(t) = U \exp(st)$ over $-\infty < t < \infty$. That is, an exponential excitation yields an exponential response at the same (complex) frequency $s = \sigma + j\omega$ as that for the input. When $\sigma = 0$, the excitation and response are both sinusoidal and the resulting response is called the **sinusoidal steady state** (SSS). Equation (9.25) shows that the SSS response is found by substituting the complex frequency $s = j\omega$ into the **transfer function**, now evaluated on complex numbers instead of differential operators as in (9.18),

$$H(s) = \frac{n_1 s + n_0}{d_1 s + d_0} \tag{9.26}$$

This transfer function represents the impulse response, $h(t)$, of which it is actually the Laplace transform, and as we found earlier, the causal impulse response is

$$h(t) = \begin{cases} \dfrac{n_0}{d_0}\delta(t) + \dfrac{n_1}{d_0}\delta'(t), & \text{if } d_1 = 0 \\[2ex] \dfrac{n_1}{d_1}\delta(t) + \left[\dfrac{d_1 n_0 - d_0 n_1}{d_1}\right] e^{-\frac{d_0}{d_1}t} 1(t), & \text{if } d_1 \neq 0 \end{cases} \tag{9.27}$$

However, practical signals are started at some finite time, normalized here to $t = 0$, instead of at $t = -\infty$, as used for the preceding exponentials. Thus, consider an input of the same type but applied only for $t > 0$; i.e., let $u(t) = U \exp(st)1(t)$. The output is found by using the convolution $y = h * u$; after a slight amount of calculation is evaluated to

$$\begin{aligned} y(t) &= h(t) * Ue^{st}1(t) \\ &= \begin{cases} H(s)Ue^{st}1(t) + \dfrac{n_1}{d_0}U\delta(t) & \text{for } d_1 = 0 \\[2ex] H(s)Ue^{st}1(t) - \dfrac{[d_1 n_0 - d_0 n_1]}{d_1 s + d_0} Ue^{-\frac{d_0}{d_1}} 1(t) & \text{for } d_1 \neq 0 \end{cases} \end{aligned} \tag{9.28}$$

For $t > 0$, the SSS remains present, while there is another term of importance when $d_1 \neq 0$. This is a transient term, which disappears after a sufficient waiting time in the case of an asymptotically stable circuit. That is, the SSS is truly a steady state, although one may have to wait for it to dominate. If a nonzero zero-input response exists, it must be added to the right side of (9.28), but for $t > 0$ this is of the same form as the transient already present, therefore, the conclusion is identical (the SSS eventually predominates over the transient terms for an asymptotically stable circuit).

Because a cosine is the real part of a complex exponential and the real part is obtained as the sum of two terms, we can use linearity of the circuit to quickly obtain the output to a cosine input when we know the output due to an exponential. We merely write the input as the sum of two complex conjugate exponentials and then take the complex conjugates of the outputs that are summed. In the case of real coefficients in the transfer function, this is equivalent to taking the real part of the output when we take the real part of the input; that is, $y = \mathcal{R}(h * u_3) = h * u$, when $u = \mathcal{R}(u_e)$, if y is real for all real u.

Network Time Constant

The time constant, t_c, was defined earlier as the time for which a transient decays to $1/e$ of the intial value. As such, the time constant shows up in signals throughout the circuit and is a very useful parameter when identifying a circuit from its responses. In an RC circuit, the time constant physically results from the interaction of the equivalent capacitor (of which only one exists in a first-order circuit) of capacitance C_{eq}, and the Thévenin's equivalent resistor, of resistance R_{eq}, that it sees. Thus, $t_c = R_{eq}C_{eq}$.

Closely related to the time constant is the **rise time**. Considering the low-pass case, the rise time, t_r is defined as the time for the unit step response to go between 10% and 90% of its final value from its initial value. This is easily calculated because the unit step response is given by

$$y_{1(\cdot)}(t) = h(t) * 1(t) = \frac{n_0}{d_0}\left[1 - e^{-\frac{d_0}{d_1}t}\right] \cdot 1(t) \tag{9.29}$$

Assuming a stable circuit and setting this equal to 0.1 and 0.9 times the final value, n_0/d_0, it is readily found that

$$t_r = \frac{d_1}{d_0} \cdot \ln(9) = [\ln(9)] \cdot t_c \approx 2.2t_c \tag{9.30}$$

At this point, it is worth noting that for theoretical studies the time constant can be normalized to 1 by normalizing the time scale. Thus, assuming d_1 and $d_0 \neq 0$ the differential equation can be written as

$$d_0 \cdot \left[\frac{d_1}{d_0} \cdot \frac{dy}{d(d_1/d_0)\left(t(d_1/d_0)\right)} + y\right] = d_0\left[\frac{dy}{dt_n} + y\right] \tag{9.31}$$

where $t_n = (d_0/d_1)t$ is the normalized time.

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9.3 Second-Order Circuits

Introduction

Because real transfer functions can be factored into real second-order transfer functions, second-order circuits are probably the most important circuits available; most designs are based upon them. As with first-order circuits, this chapter is limited to single-input-single-output linear time-invariant circuits, and unless otherwise stated, here real-valued quantities are assumed. By definition a **second-order circuit** is described by the differential equation

$$d_2 \cdot \frac{d^2y}{dt^2} + d_1 \cdot \frac{dy}{dt} + d_0 \cdot y = n_2 \cdot \frac{d^2u}{dt^2} + n_1 \cdot \frac{du}{dt} + n_0 \cdot u \tag{9.32}$$

where d_i and n_i are "denominator" and "numerator" constants, $i = 0, 1, 2$, which, unless mentioned to the contrary, are taken to be real. Continuing the notation used for first-order circuits, $y = y(\cdot)$ is the output and $u = u(\cdot)$ is the input; both u and y are generalized functions of time t. Assume that $d_2 \neq 0$, which is the normal case because any of the other special cases can be considered as cascades of real degree one circuits.

Again, treating the derivative as an operator, $p = d[\cdot]/dt$, (9.32) is written as

$$y = \frac{n_2 p^2 + n_1 p + n_0}{d_2 p^2 + d_1 p + d_0} u \tag{9.33}$$

with the **transfer function**

$$\begin{aligned} H(p) &= \frac{1}{d_2}\left[\frac{n_2 p^2 + n_1 p + n_0}{p^2 + (d_1/d_2)p + (d_0/d_2)}\right] \\ &= \frac{1}{d_2}\left[n_2 + \frac{\left(n_1 - (d_1/d_2)n_2\right)p + \left(n_0 - (d_0/d_2)n_2\right)}{p^2 + (d_1/d_2)p + (d_0/d_2)}\right] \end{aligned} \tag{9.34}$$

where the second form results by long division of the denominator into the numerator. Because they occur most frequently when second-order circuits are discussed, we rewrite the denominator in two equivalent customarily used forms:

$$p^2 + \frac{d_1}{d_2}p + \frac{d_0}{d_2} = p^2 + \frac{\omega_n}{Q}p + \omega_n^2 = p^2 + 2\zeta\omega_n p + \omega_n^2 \tag{9.35}$$

where ω_n is the undamped natural frequency ≥ 0, Q is the quality factor, and ζ is the damping factor = $1/(2Q)$. The transfer function is accordingly

$$H(p) = \frac{1}{d_2}\left[\frac{n_2 p^2 + n_1 p + n_0}{p^2 + (\omega_n/Q)p + \omega_n^2}\right] = \frac{1}{d_2}\left[\frac{n_2 p^2 + n_1 p + n_0}{p^2 + 2\zeta\omega_n p + \omega_n^2}\right] \tag{9.36}$$

Table 9.2 lists several of the more important transfer functions, which, as in the first-order case, are operators as functions of the derivative operator p.

Zero-Input and Zero-State Response

Again, as in the first-order case, a convenient tool for investigating the time-domain behavior of a second-order circuit is the **state variable description**. Letting the state vector be $x(\cdot)$, the state-space representation is

$$\begin{aligned} px &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{9.37}$$

where, as above, $p = d[\cdot]/dt$, and A, B, C, D are constant matrices. In the present case, these matrices are real and one convenient choice, among many, is

$$\begin{aligned} px &= \begin{bmatrix} 0 & 1 \\ -\dfrac{d_0}{d_2} & -\dfrac{d_1}{d_2} \end{bmatrix} x + \begin{bmatrix} n_1 - \dfrac{d_1}{d_2}n_2 \\ \left(n_0 - \dfrac{d_0}{d_2}n_2\right) - \left(n_1 - \dfrac{d_1}{d_2}n_2\right) \end{bmatrix} u \\ y &= \begin{bmatrix} \dfrac{1}{d_2} & 0 \end{bmatrix} x + \begin{bmatrix} \dfrac{n_1}{d_2} \end{bmatrix} u \end{aligned} \tag{9.38}$$

TABLE 9.2 Typical Second-Order Circuit Transfer Functions

Transfer Function	Description	Impulse Response
$\frac{n_0}{d_2}\frac{1}{p^2+2\zeta\omega_n p+\omega_n^2}$	Low-pass	$h_{lp}(t)=\frac{n_0}{d_2}\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)1(t)$
$\frac{n_2}{d_2}\frac{p^2}{p^2+2\zeta\omega_n p+\omega_n^2}$	High-pass $\theta=\arctan 2\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$	$h_{hp}(t)=\frac{n_2}{d_2}\left[\delta(t)-\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+2\theta\right)1(t)\right]$
$\frac{n_1}{d_2}\frac{p}{p^2+2\zeta\omega_n p+\omega_n^2}$	Bandpass $\theta=\arctan 2\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$	$h_{bp}(t)=\frac{n_1}{d_2}\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right)1(t)$
$\frac{n_2}{d_2}\frac{p^2+\omega_0^2}{p^2+2\zeta\omega_n p+\omega_n^2}$	Band-stop	$h_{bs}(t)=h_{hp}(t)+\frac{n_2\omega_0^2}{n_0}h_{lp}(t)$
$\frac{n_2}{d_2}\frac{p^2-2\zeta\omega_n p+\omega_n^2}{p^2+2\zeta\omega_n p+\omega_n^2}$	All-pass	$h_{ap}(t)=\frac{n_2}{d_2}\left[\delta(t)-\frac{4\zeta\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right)1(t)\right]$
$\frac{n_0}{d_2}\frac{1}{p^2+\omega_n^2}$	Oscillator,	$h_{osc}(t)=\frac{n_0}{d_2}\sin(\omega_n t)\cdot 1(t)$
	when u = 0	$y(t)\big\vert_{u=0}=y(0)\cdot\cos(\omega_n t)+\frac{y'(0)}{\omega_n}\cdot\sin(\omega_n t)$

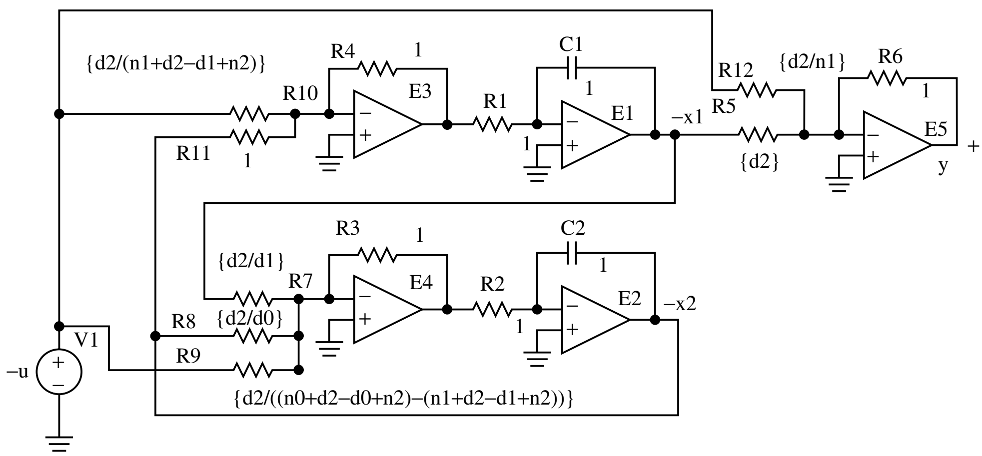

FIGURE 9.2 Generic, second-order op-amp RC circuit.

Here, the state is the 2-vector $x = [x_1\ x_2]^T$, with the superscript T denoting transpose. Normally, the state would consist of capacitor voltages and/or inductor currents, although at times one may wish to use linear combinations of these. From these state variable equations, a generic operational-amplifier (op-amp) RC circuit to realize any of this class of second-order circuits is readily designed and given in Figure 9.2. In the figure, all voltages are referenced to ground and normalized capacitor and resistor values are listed. Alternate designs in terms of only CMOS differential pairs and capacitors can also be given [3], while a number of alternate circuits exist in the catalog of Sallen and Key [4].

Because (9.38) represents a set of linear constant coefficient differential equations, superposition applies and its solution can again be broken into two parts, the part due to initial conditions, $x(0)$, called the zero-input response, and the part due solely to the input u, the zero-state response.

The **zero-input response** is readily found by solving the state equations with $u = 0$ and initial conditions $x(0)$. The result is $y(t) = C\exp(At)\,x(0)$, which can be evaluated by several means, including the following. Using a prime to designate the time derivative, first note that when $u = 0$, $x_1(t) = d_2\,y(t)$ and (from the first row of A) $x_1(t)' = x_2(t) = d_2\,y(t)'$. Thus, $x_1(0) = d_2\,y(0)$ and $x_2(0) = d_2\,y'(0)$, which allow the initial conditions to be expressed in terms of the measurable output quantities. To evaluate $\exp(At)$, note that its terms are linear combinations of terms with complex frequencies that are zeroes of the characteristic polynomial

$$\det(s1_2 - A) = \det\begin{pmatrix} s & -1 \\ \omega_n^2 & s + 2\zeta\omega_n \end{pmatrix} = s^2 + 2\zeta\omega_n s + \omega_n^2 = (s - s_-)(s - s_+) \qquad (9.39)$$

For which the roots, called **natural frequencies**, are

$$s_\pm = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n = \left(-1 \pm \sqrt{1 - 4Q^2}\right)\frac{\omega_n}{2Q} \qquad (9.40)$$

The case of equal roots will only occur when $\zeta^2 = 1$, which is the same as $Q^2 = 1/4$, for which the roots are real. Indeed, if the damping factor, ζ, is > 1 in magnitude, or equivalently, if the quality factor, Q, is $<1/2$ in magnitude, the roots are real and the circuit can be considered a cascade of two first-order circuits. Thus, assume here and in the following that unless otherwise stated, $Q^2 > 0.25$, which is the same as $\zeta^2 < 1$, in which case the roots are complex conjugates, $s_- = s_+^*$

$$s_\pm = \left(-\zeta \pm j\sqrt{1 - \zeta^2}\right)\omega_n = \left(-1 \pm j\sqrt{4Q^2 - 1}\right)\frac{\omega_n}{2Q}, \quad j = \sqrt{-1} \qquad (9.41)$$

By writing $y(t) = a \cdot \exp(s_+ t) + b \cdot \exp(s_- t)$, for unknown constants a and b, differentiating and setting $t = 0$ we can solve for a and b, and after some algebra and trigonometry obtain the zero-input response

$$y(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left\{y(0)\cdot\cos\left(\sqrt{1 - \zeta^2}\,\omega_n t - \theta\right) + \frac{y'(0)}{\omega_n}\cdot\sin\left(\sqrt{1 - \zeta^2}\,\omega_n t\right)\right\} \qquad (9.42)$$

where $\theta = \arctan2(\zeta/\sqrt{1 - \zeta^2})$ with $\arctan2(\cdot)$ being the arc tangent function that incorporates the sign of its argument.

The form given in (9.42) allows for some useful observations. Remembering that this assumes $\zeta^2 < 1$, first note that if no damping occurs, that is, $\zeta = 0$, then the natural frequencies are purely imaginary, $s_+ = j\omega_n$ and $s_- = -s_+$, and the response is purely oscillatory, taking the form shown in the last line of Table 9.2. If the damping is positive, as it would be for a passive circuit having some loss, usually via positive resistors, then the natural frequencies lie in the left half s-plane, and y decays to zero at positive infinite time so that any transients in the circuit die out after a sufficient wait. The circuit is then called **asymptotically stable**. However, if the damping is negative, as it could be for some positive feedback circuits or those with negative resistance, then the response to nonzero initial conditions increases in amplitude without bound, although in an oscillatory manner, as time increases, and the circuit is said to be **unstable**. In the unstable case, as time decreases through negative time the amplitude also damps out to zero, but usually the responses backward in time are not of as much interest as those forward in time.

For the **zero-state response**, the impulse response, $h(t)$, is convoluted with the input, that is, $y = h * u$, for which we can use the fact that $h(t)$ is the inverse Laplace transform of $H(s) = C[sI_2 - A]^{-1}B$. The denominator of $H(s)$ is $\det(sI_2 - A) = s^2 + 2\zeta\omega_n s + \omega_n^2$, for which the causal inverse Laplace transform is

$$e^{s_+t}1(t) * e^{s_-t}1(t) = \begin{cases} \dfrac{e^{s_+t} - e^{s_-t}}{s_+ - s_-}1(t) & \text{if } s_- \neq s_+ \\ te^{s_+t}1(t) & \text{if } s_- = s_+ \end{cases} \tag{9.43}$$

Here, the bottom case is ruled out when only complex natural frequencies are considered, following the assumption of handling real natural frequencies in first-order circuits, made previously. Consequently,

$$e^{s_+t}1(t) * e^{s_-t}1(t) = \frac{e^{s_+t} - e^{s_-t}}{s_+ - s_-}1(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\omega_n t\right)\cdot 1(t) \tag{9.44}$$

Again, assuming $\zeta^2 < 1$ using the preceding calculations give the zero-state response as

$$\begin{aligned} y(t) &= \frac{1}{d_2}\left\{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\omega_n t\right)1(t) * \right. \\ &\quad \left.\left[\left(n_1 - \frac{d_1}{d_2}n_2\right)\delta'(t) + \left(n_0 - \frac{d_0}{d_2}n_2\right)\delta(t)\right] + n_2\delta(t)\right\} * u(t) \\ &= \frac{1}{d_2}\left\{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\omega_n t\right)1(t) * \right. \\ &\quad \left.\left[n_2\delta''(t) + n_1\delta'(t) + n_0\delta(t)\right]\right\} * u(t) \end{aligned} \tag{9.45}$$

The bottom equivalent form is easily seen to result from writing the transfer function $H(p)$ as the product of two terms $1/[d_2(p^2 + 2\zeta\omega_n p + \omega_n^2)$ and $[n_2p^2 + n_1p + n_0]$ convoluting the causal impulse response (the inverse of the left half-plane converging Laplace transform), of each term. From (9.45), we directly read the **impulse response** to be

$$\begin{aligned} h(t) = \frac{1}{d_2}&\left\{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\omega_n t\right)1(t)\right. \\ &\left. * \left[n_2\delta''(t) + n_1\delta'(t) + n_0\delta(t)\right]\right\} \end{aligned} \tag{9.46}$$

Equations (9.45) and (9.46) are readily evaluated further by noting that the convolution of a function with the second derivative of the impulse, the first derivative of the impulse, and the impulse itself is the second derivative of the function, the first derivative of the function, and the function itself, respectively. For example, in the low-pass case we find the impulse response to be, using (9.46),

$$h_{lp}(t) = \frac{n_0}{d_2}\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\omega_n}\sin\left(\sqrt{1-\zeta^2}\omega_n t\right)1(t) \tag{9.47}$$

By differentiating we find the bandpass and then high-pass impulse responses to be, respectively,

$$h_{bp}(t) = \frac{n_1}{d_2} \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) 1(t) \tag{9.48}$$

$$h_{hp}(t) = \frac{n_2}{d_2}\left[\delta(t) - \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + 2\theta\right) 1(t)\right] \tag{9.49}$$

In both cases, the added phase angle is given, as in the zero input response, via $\theta = \text{arctan2}(\zeta/\sqrt{1-\zeta^2})$. By adding these last three impulse responses suitably scaled the impulse responses of the more general second-order circuits are obtained.

Some comments on **normalizations** are worth mentioning in passing. Because $d_2 \neq 0$, one could assume d_2 to be 1 by absorbing its actual value in the transfer function numerator coefficients. If $\omega_n \neq 0$, time could also be scaled so that $\omega_n = 1$ could be taken, in which case a normalized time, t_n, is introduced. Thus, $t = \omega_n t_n$ and, along with normalized time, comes a normalized differential operator $p_n = d[\cdot]/dt_n = d[\cdot]/d(t/\omega_n) = \omega_n p$. This, in turn, leads to a normalized transfer function by substituting $p = p_n/\omega_n$ into $H(p)$. Thus, much of the treatment could be carried out on the normalized transfer function x

$$H_n(p_n) = H(p) = \frac{n_{2n}p_n^2 + n_{1n}p_n + n_{0n}}{p_n^2 + 2\zeta p_n + 1} \qquad p_n = \omega_n p \tag{9.50}$$

In this normalized form, it appears that the most important parameter in fixing the form of the response is the damping factor $\zeta = 1/(2Q)$.

Transient and Steady-State Responses

Let us now excite the circuit with an eternal exponential input, $u(t) = U\exp(st)$ for $-\infty < t < \infty$ at the complex frequency $s = \sigma + j\omega$, where s is chosen as different from either of the natural frequencies, $s_\pm$, and U is a constant. As with the first-order and, indeed, any higher-order, case the response is $y(t) = Y(s)\exp(st)$, as is observed by direct substitution into (9.32). This substitution yields directly

$$Y(s) = \frac{1}{d_2}\left[\frac{n_2 s^2 + n_1 s + n_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]\cdot U \tag{9.51}$$

where $y(t) = Y(s)\exp(st)$ for $u(t) = U\exp(st)$ over $-\infty < t < \infty$. That is, an exponential excitation yields an exponential response at the same (complex) frequency $s = \sigma + j\omega$ as that for the input, as long as s is not one of the two natural frequencies. (s may have positive as well as negative real parts and is best considered as a frequency and not as the Laplace transform variable because the latter is limited to regions of convergence.) Because the denominator polynomial of $Y(\dot{s})$ has roots which are the natural frequencies, the magnitude of Y becomes infinite as the frequency of the excitation approaches s_+ or s_-. Thus, the natural frequencies s_+ and s_- are also called **poles** of the transfer function.

When $\sigma = 0$ the excitation and response are both sinusoidal and the resulting response is called the **sinusoidal steady state** (SSS). From (9.51), the SSS response is found by substituting the complex frequency $s = j\omega$ into the transfer function, now evaluated on complex numbers rather than differential operators as above,

$$H(s) = \frac{1}{d_2}\left[\frac{n_2 s^2 + n_1 s + n_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \tag{9.52}$$

Next, an exponential input is applied, which starts at $t = 0$ instead of at $t = -\infty$; i.e., $u(t) = U\exp(st)1(t)$. Then, the output is found by using the convolution $y = h * u$, which, from the discussion at (9.45), is expressed as

$$y(t) = h * u = \frac{1}{d_2} e^{s_+ t} 1(t) * e^{s_- t} 1(t) * \left[n_2 \delta''(t) + n_1 \delta'(t) + n_0 \delta(t) \right] * e^{st} 1(t)$$

$$= H(s) U e^{st} 1(t) + \left\{ \frac{1}{d_2 (s_+ - s_-)} \left[\left(\frac{N(s)}{s_+ - s} + n_2 (s + s_+) + n_1 \right) e^{s_+ t} \right. \right. \tag{9.53}$$

$$\left. \left. - \left(\frac{N(s)}{s_+ - s} + n_2 (s + s_-) + n_1 \right) e^{s_- t} \right] 1(t) \right\}$$

in which $N(s)$ is the numerator of the transfer function and we have assumed that s is not equal to a natural frequency. The second term on the right within the braces varies at the natural frequencies and as such is called the **transient response**, while the first term is the term resulting directly from an eternal exponential, but now with the negative time portion of the response removed. If the system is stable, the transient response decays to zero as time increases and, thus, if we wait long enough the transient response of a stable system can be ignored if the complex frequency of the input exponential has a real part that is greater than that of the natural frequencies. Such is the case for exponentials that yield sinusoids; in that case $\sigma = 0$, or $s = j\omega$. In other words, for an asymptotically stable circuit the output approaches that of the SSS when the input frequency is purely imaginary. If we were to excite at a natural frequency then the first part of (9.53) still could be evaluated using the time-multiplied exponential of (9.43); however, the transient and the steady state are now mixed, both being at the same "frequency."

Because actual sinusoidal signals are real, we use superposition and the fact that the real part of a complex signal is given by adding complex conjugate terms:

$$\cos(\omega t) = \Re\left[e^{j\omega t}\right] = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{9.54}$$

This leads to the SSS response for an asymptotically stable circuit excited by $u(t) = U \cos(\omega t) 1(t)$ to be

$$y(t) = \frac{H(j\omega) U e^{j\omega t} + H(-j\omega) U^* e^{-j\omega t}}{2}$$
$$= |H(j\omega)||U| \cos(\omega t + \angle H(j\omega) + \angle U) \tag{9.55}$$

Here, we assumed that the circuit has real-valued components such that $H(-j\omega)$ is the complex conjugate of $H(j\omega)$. In which case, the second term in the middle expression is the complex conjugate of the first.

Network Characterization

Although the impulse response is useful for theoretical studies, it is difficult to observe it experimentally due to the impossibility of creating an impulse. However, the unit step response is readily measured, and from it the impulse response actually can be obtained by numerical differentiation if needed. However, it is more convenient to work directly with the unit step response and, consequently, practical characterizations can be based upon it. The treatment most conveniently proceeds from the normalized low-pass transfer function

$$H(p) = \frac{1}{p^2 + 2\zeta p + 1}, \quad 0 < \zeta < 1 \tag{9.56}$$

The **unit step response** follows by applying the input $u(t) = 1(t)$ and noting that the unit step is the special case of an exponential multiplied unit step, where the frequency of the exponential is zero. Conveniently, (9.43) can be used to obtain

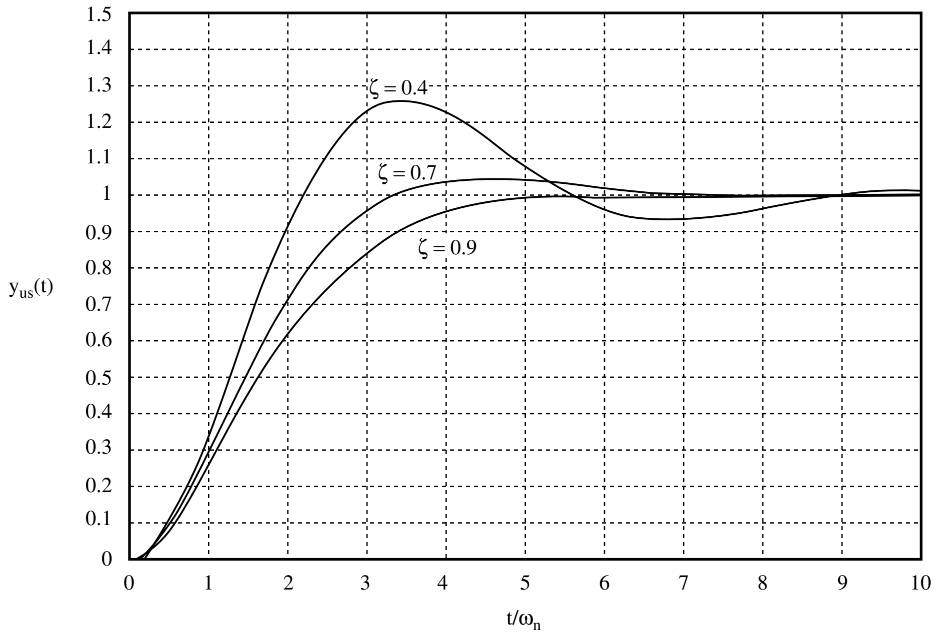

FIGURE 9.3 Unit step response for different damping factors.

$$y_{us}(t) = 1(t) - \frac{e^{-\zeta t}}{\sqrt{1-\zeta^2}} \cos\left(\sqrt{1-\zeta^2}\, t - \theta\right) \cdot 1(t), \quad \theta = \arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{9.57}$$

Typical unit step responses are plotted in Figure 9.3 where, for a small damping factor, overshoot can be considerable, with oscillations around the final value and in addition, a long settling time before reaching the final value. In contrast, with a large damping factor, although no overshoot or oscillation occurs, the rise to the final value is long. A compromise for obtaining a quick rise to the final value with no oscillations is given by choosing a damping factor of 0.7, this being called the **critical value**; i.e., **critical damping** is $\zeta_{crit} = 0.7$.

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